Translation of the Kimberling's Glossary into barycentrics
(v48-dvipdfm)

June 13, 2012

Acknowledgement. This document began its life as a private copy of the Glossary accompanying the Encyclopedia of Clark Kimberling. This Glossary - as its name suggests - is organized alphabetically. As a –never satisfied– newcomer, I would have preferred a progression from the easiest to the hardest topics and I reordered this document in my own way. It is unclear whether this new ordering will be useful to someone else ! In any case a detailed index is provided.

Second point, the Glossary is written using trilinear coordinates. From an advanced point of view, these coordinates are neither better nor worse than the barycentric coordinates. Nevertheless, having some practice of the barycentrics, and none of the trilinears, I undertook to translate everything, from one system to another. In any case, this was a formative exercise, and this also puts the focus on the covariance/contravariance properties that were subsequently systematized.

Drawings are the third point. Everybody knows –or should know– that geometry is not possible at all without drawings. Having no intention to pay royalties for using rulers and compasses, I turned to an open source software (kseg) in order to produce my own drawings. Thereafter, I have used Geogebra, together with pstricks. What a battle, but no progress without practice !

Subsequently, other elements have been incorporated from other sources, including materials about cubics, from the Gibert web site. Finally, original elements were also added. As it will appear at first sight, "pldx" is addicted to a tradition that requires a precisely specified universal space for each object to live in.

A second massive addiction of the author is computer algebra. Having at your fingertips a tool that gives the right answer to each and every expansion or factorization, and never lost the small paper sheet where the computation of the week was summarized is really great. Moreover, being constrained to explain everything to a computer helps to specify all the required details. For example, the "equivalence up to a proportionality factor" doesn't apply in the same way to a matrix whether this matrix describes a collineation, a triangle, a trigon or a set of incidence relations.

In this document, "beautiful geometrical proofs" are avoided as most as possible, since they are the most error prone. A safe proof of "*the triangle of contacts of the inscribed circle and the triangle of the mid-arcs on the circumcircle admit the similicenter of the two circles as perspector (and therefore as center of similitude)*" is :

`ency(persp(matcev(pX(7)),matucev(pX(1))))` $\mapsto$ `56`

where the crucial point is `ency`, i.e. a safe implementation of the Kimberling's search key method to explore the database.

To summarize, the present document is rather a "derived work", where the elements presented are not intended to be genuine, apart perhaps from the way to assemble the pieces and cook them all together.

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Chapter 1

Introduction

Many changes have occurred in the way we are doing geometry, from the old ancient times of Euclid and Apollonius. Most of them are related to yet another way of performing "automated" computing of properties, rather than relying on intuition to find "beautiful geometric proofs". Many individuals have contributed to this long process, and attributing a given discovery to a given individual is not an easy task (Coolidge, 1940). In fact, the emerging milestones in this long road are rather the individuals that have summarized the discoveries of their time in an efficient way of writing the questions to solve, so that the writing by itself "thinks for you" and, under a least action trend, leads you towards the required result.

Writing numbers and their computations in a tractable manner is associated with Al-Khwarizmi and his *Algebra* (825). Using coordinates (x, y) to describe points and compute their geometrical properties (as well as the exponent notation for polynomials) is associated with Descartes and his *Géometrie* (1637). Using homogeneous coordinates $x : y : z$ to implement the principle of continuity when dealing with objects escaping to infinity is associated with Moebius and his *barizentrische Calcul* (1827).

More recently, the very idea of stamping an hashcode on each noteworthy point involved in Triangle Geometry and then practice some kind of computer aided inventory management (Kimberling, 1998-2010) has changed the practice of geometers. This idea has emerged from a more general trend, where

> barycentrics are understood to ***define*** points, lines, circles, triangle centers, etc., and zero determinants are understood to ***define*** collinearity and concurrence. Doing that way, triangle geometry, formally speaking, is much more general than the study of a single Euclidean triangle. In the formal treatment, sometimes called *transfigured triangle geometry*, the symbols a, b, c are regarded as algebraic unknowns, so that points, defined as functions of a, b, c, are not the usual points of a two-dimensional plane. When (a, b, c) are real numbers restricted by the "triangle inequalities" for sidelengths, the resulting geometry is traditional triangle geometry (Kimberling, 1998).

When possible, computed proofs are given that use formal computing tools. This kind of proof is deprecated by several authors. Nevertheless, these proofs are the easiest since all the messy job is done by a computer and are also the safest. A construction that sounds like a "beautiful geometrical proof" is too often invalid due to some hidden exception. During a computerized proof, exceptions are appearing as multiplicative factors, according to the polynomial model :

$$\textit{conclusion} \times \textit{exceptions} = \textit{hypothesis}$$

To quote the Knuth's foreword to Petkovsek et al. (1996) :

> Science is what we understand well enough to explain to a computer. Art is everything else we do. During the past several years important parts of mathematics has been transformed from an Art to a Science.

1.1 Special remarks for French natives

Commençons par deux faux amis. Le terme US "line" désigne une "droite" (opposée à une courbe), tandis qu'une "ligne" (opposée à une colonne) est désignée par le terme US "row". Ensuite, une

remarque sur les différences de "feeling" vis à vis des livres scientifiques. Lorsque vous écrivez pour un public américain, mieux vaut commencer par montrer que le sujet est suffisamment intéressant pour mériter du temps et de la peine. Lorsque vous écrivez pour un public français, mieux vaut commencer par montrer que l'auteur dispose d'une hauteur de vue suffisante. Si vous sentez les choses de cette façon, la Section **??** est le bon endroit par où commencer.

1.2 Basic objects : points and lines

Abbreviation etc. will stand for "et cyclically". After an "A-object" has been defined as $F(A, B, C)$, then the B-object is $F(B, C, A)$, obtained by the cyclic permutation $ABC \mapsto BCA$, and the C-object is, likewise, $F(C, A, B)$. Example: If A' is the point where lines AP and BC meet, and B' and C' are defined cyclically, then B' is where lines BP and CA meet, and C' is where lines CP and AB meet.

Definition 1.2.1. A **point** is an element of $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$. To tell the same thing more simply, a point is represented by a **column** of three numbers (the barycentrics of the point), not all of them being zero, and such a column is dealt "in a projective manner", i.e. up to a proportionality factor. An efficient way to write such a projective column is the colon notation :

$$P = p : q : r \qquad \text{meaning that} \qquad P \simeq \begin{pmatrix} p \\ q \\ r \end{pmatrix} \simeq \begin{pmatrix} k\,p \\ k\,q \\ k\,r \end{pmatrix} \forall k \in \mathbb{R} \setminus \{0\}$$

Definition 1.2.2. A **line** is an element of the dual of the point space. To tell the same thing more simply, a line is represented by a **row** of three numbers (the barycentrics of the line), not all of them being zero, and such a row is dealt "in a projective manner", i.e. up to a proportionality factor. A line will be described as :

$$\Delta \simeq \begin{pmatrix} \rho & \sigma & \tau \end{pmatrix} \simeq \begin{pmatrix} \lambda\rho & \lambda\sigma & \lambda\tau \end{pmatrix} \forall \lambda \in \mathbb{R} \setminus \{0\}$$

Notation 1.2.3. In all these definitions, property $p^2 + q^2 + r^2 \neq 0$ and $\rho^2 + \sigma^2 + \tau^2 \neq 0$ are ever intended. Colon notation will *ever* be restricted to inline equations describing columns, and *never* be used for rows. Anyway, such a notation would be hopeless when dealing with matrices. We will use instead the $\simeq$ sign to denote proportionality. Therefore, except from expressions written in colons, the equal sign will ever indicate a strong, component-wise equality.

Definition 1.2.4. Incidence relations. We will say that a line $\Delta \simeq (p, q, r)$ contains a point $U = u : v : w$, or that Δ goes through U, or that U belongs to Δ when their dot product vanishes, i.e. :

$$U \in \Delta \Longleftrightarrow pu + qv + rw = 0$$

Remark 1.2.5. It is clear that incidence relation is projective, i.e. holds for any choice of the proportionality factors.

Definition 1.2.6. The **collinearity** of three points $P = p : q : r$, $U = u : v : w$, $X = x : y : z$ is defined by the following determinant equation :

$$\begin{vmatrix} x & p & u \\ y & q & v \\ z & r & w \end{vmatrix} = 0 \tag{1.1}$$

When $P \neq U$, the set of all the X that satisfies (1.1) is what is usually called the line PU.

Definition 1.2.7. A **triangle** is an ordered set of three non collinear points. Its natural representation is an invertible square "matrix of columns", where each column is defined up to a proportionality factor (right action of a diagonal matrix). On the contrary, a "may be degenerate triangle" is a matrix such that (i) columns are not proportional to each other and (ii) rank is at least two. When at least two points are equal, the triangle is "totally degenerate".

Remark 1.2.8. Without explicit permission, a triangle is not allowed to be degenerate, while totally degenerate triangles are (quite ever) to be avoided.

Definition 1.2.9. The **concurrence** of three lines $\Delta_1 \simeq (d, e, f)$, $\Delta_2 \simeq (p, q, r)$ and $\Delta_3 \simeq (u, v, w)$ is defined by the following determinant equation :

$$\begin{vmatrix} d & e & f \\ r & s & t \\ u & v & w \end{vmatrix} = 0 \tag{1.2}$$

When $\Delta_1 \neq \Delta_2$, the set of all the Δ_3 that satisfies (1.2) is usually called the pencil generated by the two lines.

Definition 1.2.10. A **trigon** is a set of ordered three non concurrent lines. Its natural representation is an invertible square "matrix of row", where each row is defined up to a proportionality factor (left action of a diagonal matrix).

Proposition 1.2.11. *The reciprocal matrix of a triangle is a trigon and conversely. Adjoint matrices can be used instead of inverses due to proportionality. Relation $\mathcal{T}^{-1}.\mathcal{T} = Id$ is nothing but the incidence relations : $A' \notin B'C'$, $A' \in A'B'$, $A' \in A'C'$ and cyclically.*

Remark 1.2.12. It should be noticed that a tetra-angle defines an hexa-gone, while an tetra-gone (quadrilateral) defines an hexa-angle : $n = n\,(n-1)\,/2$ holds only when $n = 3$.

Definition 1.2.13. The **line at infinity** $\mathcal{L}_\infty$ is the locus of points $x : y : z$ such that $x+y+z = 0$, so that any point out of $\mathcal{L}_\infty$ can be described by a triple such that $x + y + z = 1$. Using only this representation would discard $\mathcal{L}_\infty$ and is nothing but the usual affine geometry.

Definition 1.2.14. Barycentric basis. Special points $A = 1 : 0 : 0$, $B = 0 : 1 : 0$, $C = 0 : 0 : 1$ are usually identified with the vertices of a triangle in the Euclidean[1] plane, so that all (barycentric) points can be mapped onto the Euclidean plane (completed with the appropriate line at infinity). The side lengths of this triangle are denoted $BC = a$, $CA = b$ and $AB = c$. Since a triangle is not a twoangle, none of the a, b, c are allowed to vanish.

1.3 Search key

Definition 1.3.1. Standardized barycentrics are defined as follows :

$$\begin{aligned} x : y : z \notin \mathcal{L}_\infty \quad &\mapsto \quad (x, y, z) \times \frac{1}{x+y+z} \\ x : y : z \in \mathcal{L}_\infty \quad &\mapsto \quad (x, y, z) \times \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \end{aligned}$$

Exercise 1.3.2. Both normalizing factor don't vanish at the same time, except from points :

$$-1 \pm i\sqrt{3} : -1 \mp i\sqrt{3} : 2$$

Determine objects that contains these points.

Definition 1.3.3. The **Kimberling's search key** is defined as follows. Triangle $a = 6$, $b = 9$, $c = 13$ is used and then barycentrics are converted according to :

$$\begin{aligned} x : y : z \notin \mathcal{L}_\infty,\ key \quad &= \quad \frac{x}{x+y+z} \times \frac{bc}{2R} \\ x : y : z \in \mathcal{L}_\infty,\ key \quad &= \quad \frac{x}{a} \times \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right) \end{aligned} \tag{1.3}$$

Remark 1.3.4. When $X \in \mathcal{L}_\infty$, $key\,(X)$ is the first normalized trilinear. Otherwise $key\,(X)$ is simply the directed distance between X and sideline BC of the reference triangle. The actual value of factor $bc \div 2R$ is $4\sqrt{35}/3$. Reference values of this search key are provided by the ETC (Kimberling, 1998-2010).

[1] This kind of plane would be better called euclidian rather than Euclidean, since the very idea to compute figures instead of drawing computations is as far as possible of the thoughts and practices of the writer of the celebrated *Elements*.

Definition 1.3.5. The **Morley's search key** is the search key associated with a point defined by its Morley affix. Details and formula are provided in the corresponding chapter. See Proposition 12.3.15.

Example 1.3.6. Key conflicts can occur. For example, using $a = 6$, $b = 9$, $c = 13$ leads to :

$$\begin{aligned} X_{667} &= -288 : 1701 : -2197 \\ isc\,(X_{1319}) &= 252 : 219 : 215 \end{aligned}$$

where isc is the inverse in Spieker circle. Both points have the same search value, namely 18/49.

Remark 1.3.7. In 2009/08/27, the minimal distance between two search keys was $4.8\,10^{-7}$. This can only become smaller as more points are added to the database. Therefore, one has to be careful when computing a search key, in order to face the possibility of huge cancelling terms. Using Maple `Digits:=20` seems to be safe.

1.4 Figures

1.4.1 Using kseg

In this document, figures were initialy drawn using ***kseg*** (Baran, 1999-2006). Indeed, the very idea to pay something for using Pythagoras' theorem seems terrific. The only two problems encountered :

1. While figures are fine in the Linux version, constructions are badly saved. Use Linux version for figures, and win$ version (through wine) for the macros.

2. In kseg, postscript figures are intended to be landscape... while your favorite lyx+tex+xdvi configuration hate rotating. You can play with the portrait option (in kseg) but this will not work. Use a script to :

   - replace "90 rotate" by "0 794 translate" and "Landscape" by "Portrait" in the *.ps file
   - convert *.ps to *.eps... using `ps2eps -f`
   - change the bounding box g+40, d+310
   - anyway, you have to act on the bounding box

1.4.2 Using geogebra

Due to an increasing community using geogebra, some figures are drawn using this tool.

1. The best graphical output is obtained as a *.pdf file. The result is a page, that must be reduced to its bounding box by a tool like `pdfcrop`.

2. Obtening the second intersection of two objects requires some cryptic incantations :
```
LListe1={Intersect[object1, object2]}"
Liste1=RemoveUndefined[ "Sequence[If[Element[LListe1, i] = M,
     (NaN, NaN), Element[LListe1, i]], i, 1.0, 2.0]
xx=Element[Liste1,1]
```

3. Dealing with macros is not easy. Most of the time, you have to unzip the *.ggb file and modify it with a text editor.

1.5 Type-keeping and type-crossing functions

Remark 1.5.1. The type-keeping/type-crossing properties are better understood when they are described in terms of collineations. Chapter 13 will be devoted to this topic. The aim of the current Section is only to provide some useful tools as soon as possible.

Definition 1.5.2. Trilinears and **barycentrics**. Triangle people splits into a barycentric tribe and a trilinear tribe. The trilinear tribe thinks that trilinears, i.e. $p : q : r$[2] are better looking than barycentrics and redefine everything according to their preferences. The barycentric tribe thinks that barycentrics, i.e. $p : q : r$[3] are better looking than trilinears and redefine everything according to their preferences.

Remark 1.5.3. Trilinears can be measured directly on the figure, since they are the directed distances to the sidelines. When compasses were actual compasses and not a button to click over, using trilinears was a must. Nowadays, the existence –and the persistence– of both systems can be used for an interesting renewal of the Capulet against Montaigu story, as in `http://mathforum.org/kb/message.jspa?messageID=1091956`. But this could also be used to gain a better insight over many point-transforms used in the Triangle Geometry.

Definition 1.5.4. Vectors are covariant, while forms are contravariant. Therefore, coordinates that measure a vector are forms and therefore contravariant. At the same time, coordinates that measure a form are covariant. This is $p : q : r$ is contravariant, while $[p, q, r]$ is covariant.

Definition 1.5.5. We will say that a function $P \mapsto f(P) \, : \, p : q : r \mapsto u : v : w$ is type-keeping or type-crossing or type scrambling according to :

$$\begin{cases} \text{type keeping} & when \quad f(\alpha p : \beta q : \gamma r) = \alpha u : \beta v : \gamma w \\ \text{type crossing} & when \quad f(\alpha p : \beta q : \gamma r) = \frac{u}{\alpha} : \frac{v}{\beta} : \frac{w}{\gamma} \\ \text{type scrambling} & \text{otherwise} \end{cases}$$

For a function of several variables, global type-keeping means :

$$f(\alpha p{:}\beta q{:}\gamma r,\, \alpha u{:}\beta v{:}\gamma w) = \alpha x{:}\beta y{:}\gamma z \quad when \quad f(p{:}q{:}r,\, u{:}v{:}w) = x{:}y{:}z$$

Remark 1.5.6. An object that is intended to describe a point has to be contravariant. An object that is intended to describe a line has to be covariant, while relationships like collinearity (1.1) and concurrence(1.2) have to be invariant. Therefore a function whose input and output are points has to be type-keeping. In the same way, a function whose input and output are lines has to be type-keeping. On the contrary, a function whose entries are points and output are lines has to be type-crossing. In the same way, a function whose entries are lines and output are points has to be type-crossing. These facts are the reasons why both tribes, using barycentrics or trilinears, are proceeding to the same geometry.

Definition 1.5.7. Barycentric multiplication is the multiplication component by component of the barycentrics of two points. This operation is denoted :

$$P *_b X = p\,x : q\,y : r\,z \tag{1.4}$$

Component-wise multiplication of trilinears would be another possibility. Trilinear people acts like that.

Definition 1.5.8. Barycentric division is the division component by component of the barycentrics of two points. This operation is denoted :

$$P \div_b X = \frac{p}{x} : \frac{q}{y} : \frac{r}{z} \tag{1.5}$$

Component-wise division of trilinears would be another possibility. Trilinear people acts like that.

Remark 1.5.9. These transforms are introduced here to provide an easy description of some other transforms. The study of their geometrical meaning ist postponed to Chapter 14. For our present needs, we only need to remark that both :

$$X \mapsto X *_b P \div_b U \quad and \quad X \mapsto P *_b U \div_b X$$

are globally type-keeping transforms, that can be used to obtain points from points, or lines from lines.

[2] someone from the barycentric tribe would write them $p/a : q/b : r/c$, since she would use $p : q : r$ for the barycentrics

[3] someone from the trilinear tribe would write them $ap : bq : cr$, since she would use $p : q : r$ for the trilinears

Definition 1.5.10. Sqrtdiv. Let $F = f : g : h$ be a fixed point, and U a moving point, restricted to avoid the sidelines of ABC. The mapping defined by :

$$sqrtdiv_F\,(U) \doteq U_F^{\#} \doteq \frac{f^2}{u} : \frac{g^2}{v} : \frac{h^2}{w} \tag{1.6}$$

is globally type-keeping and describes a pointwise transform, whose fixed points are the four $\pm f : \pm g : \pm h$. This map $U \mapsto U_F^{\#}$ is exactly the same as $U \mapsto U_P^{*}$ defined by $U_P^* = P \div_b U$ and $P \doteq F *_b F$. The second form is often used, introducing a fictitious point $P = f^2 : g^2 : h^2$.

Remark 1.5.11. Using # instead of $*$ in this context is yet been done by cubics' people (Ehrmann and Gibert, 2009) : fixed points of the transform (F and its relatives) have a clearer geometrical meaning than P. On the other hand, when P crosses the borders of ABC, point F will have imaginary coordinates : not so visual configuration.

Remark 1.5.12. The converse operation of *sqrtdiv* would be *sqrtmul* defined as $(U, X) \mapsto \sqrt{ux} : \sqrt{vy} : \sqrt{wz}$ but this map is multi-defined. When U, X are triangle centers and ux, vy, wz are perfect squares, it makes sense to fix signs so that F is also a triangle center.

1.6 Duality between point and lines

Does equation $pu + qv + rw = 0$ means $P \in \Delta_U$ or $U \in \Delta_P$? Without any further indication, one cannot decide which is the point and which is the line. This is called duality. If you want to be specific, you have to say :

$$\begin{bmatrix} u & v & w \end{bmatrix} . \begin{bmatrix} p \\ q \\ r \end{bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} p & q & r \end{bmatrix} . \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0$$

and remember how points/lines are mapped into columns/rows. In any case, points aren't lines and columns aren't rows. An efficient formulation of incidence axioms must recognize this elementary fact.

Definition 1.6.1. The **wedge** operator is the universal factorization of the determinant. This means that wedge of two columns is a row, while wedge of two rows is a column. One has :

$$\begin{aligned} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \wedge \begin{pmatrix} u \\ v \\ w \end{pmatrix} &\simeq (qw - rv,\ ru - pw,\ pv - qu) \\ (p, q, r) \wedge (u, v, w) &\simeq \begin{pmatrix} qw - rv \\ ru - pw \\ pv - qu \end{pmatrix} \end{aligned}$$

Proposition 1.6.2. *When $P \neq U$, the barycentrics of the line PU are provided by operation $P \wedge U$. As it should be, this operation is commutative and is type-crossing.*

Proof. The wedge of two points cannot be $0 : 0 : 0$ when the points are different, therefore $\Delta = P \wedge U$ defines a line. By definition, we have :

$$\left(\begin{pmatrix} p \\ q \\ r \end{pmatrix} \wedge \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right) . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{vmatrix} p & u & x \\ q & v & y \\ r & w & z \end{vmatrix}$$

and the conclusion comes from the fact that inclusion of a line into another implies equality. type-crossing is obvious for stratospherical reasons... and can be checked on the components (up to a global $\alpha\beta\gamma$ factor). When dealing with lines, the same reasonment shows that "line wedge line" is a point. □

Proposition 1.6.3. *When line Δ_{12} is given by points P_1 and P_2 (with $P_1 \neq P_2$) and line Δ_3 is given by its barycentrics then either both lines are equal or their intersection M is given by :*

$$\Delta_{12} \cap \Delta_3 \simeq (\Delta_3 \,.\, P_1)\, P_2 - (\Delta_3 \,.\, P_2)\, P_1$$

Proof. Call M this object. It is clear that $M \in P_1P_2$. And we can check that $\Delta_1 \cdot M = 0$. Another proof is that $\simeq$ is in fact a componentwise identity. □

Proposition 1.6.4. *Suppose that lines Δ_{12} and Δ_{34} are respectively defined by points P_1, P_2 and points P_3, P_4. Then either both lines are equal or their intersection M is given by :*

$$M \doteq (P_1 \wedge P_2) \wedge (P_3 \wedge P_4) \simeq P_2 \det[P_1P_3P_4] - P_1 \det[P_2P_3P_4]$$

Proof. Obvious from the previous proposition and the definition $\det[P_1P_3P_4] = (P_3 \wedge P_4) \cdot P_1$. Another proof is that $\simeq$ is in fact a componentwise identity. □

Definition 1.6.5. The **wedge point** $X_\wedge$ **of a line** is what is obtained by a simple transposition the barycentrics. This way of doing is based on a misperception of the wedge operation since $PU = (P \wedge U)$ is a line (row) and not a point (column). When written in trilinears, this object don't look good. Not without reason.

Definition 1.6.6. The **Weisstein point** X_W **of a line** is what is obtained when applying the same misperception to trilinears. Applied to line PU, this leads to the **crossdifference** of P and U. When written in barycentrics :

$$X_W = \text{crossdiff}(P, U) = a^2(qw - rv) : b^2(ru - pw) : c^2(pv - qu)$$

this object don't look good. Not without reason.

A better founded concept must lead to a type-crossing transform.

Definition 1.6.7. The **tripole** of a line and the **tripolar** of a point is what is obtained by "transpose and reciprocate". Clearly, the one-to-one correspondence between pole and polar is lost when a coordinate vanishes (line through a vertex, or point on a sideline).

Remark 1.6.8. A less stratospheric definition of the tripolar is given in Definition 3.4.2.

Remark 1.6.9. When applying "transpose and reciprocate", both tribes are thinking they are acting "their way", and are talking about "trilinear pole" and "barycentric pole". But the result is the same since reciprocation of barycentrics (aka isotomic conjugacy, Section 3.3) acts over $X_\wedge$ while reciprocation of trilinears (aka isogonal conjugacy, Section 6.2) acts over X_W. To summarize (using later introduced concepts) :

$$X_\Lambda = {}^t\Delta \ ; \ X_W = {}^t\Delta *_b X_6 \ ; \ \text{tripole} = isotom(X_\Lambda) = isogon(X_K) \tag{1.7}$$

Remark 1.6.10. When tripole is at infinity, the line is tangent to the Steiner in-ellipse (cf Example 9.6.1).

Example 1.6.11. Table 1.1 describes some well known lines. For example, the Euler line goes through X_2(centroid) and X_3(circumcenter) . Its equation is $\sum x\,(b^2 - c^2)(b^2 + c^2 - a^2) = 0$. Formally, center $(b^2 - c^2)(b^2 + c^2 - a^2)$ is $X_{525} = {}^t\Delta$, while center $a^2\,(b^2 - c^2)(b^2 + c^2 - a^2)$ is $X_{647} = X_W$. This center has been used sometimes to describe lines, leading to $Euler = L_{647}$. The next column (∞) gives the infinity point while the remaining two columns give the later defined orthopoint ($\infty^\perp$) and orthopole ($\perp$pole).

Proposition 1.6.12. *Tripole and tripolar, being correctly typed, are constructible (Figure 1.1). Start from P. Draw AP and obtain $A' = AP \cap BC$. Construct $A'' \in BC$ so that division $BCA'A''$ is harmonic (Section 3.2). Act cyclically and obtain B'' and C''. Then $A''B''C''$ are collinear, and the line they define is nothing but the tripolar of P. (and are named* tripo *in Table 1.1).*

Remark 1.6.13. One of the most important consequence of all these duality formulas is the rock-solid equality giving the intersection of two lines each of them defined by two points :

$$(PQ \cap RS) = (P \wedge Q) \wedge (R \wedge S)$$

name	line	tripo	${}^t\Delta$	L	∞	$\infty^\perp$	$\perp$po	#
Euler	X_2-X_3	X_{648}	X_{525}	L_{647}	X_{30}	X_{523}	X_{125}	404
Infinity	X_{30}-X_{511}	X_2	X_2	L_6				269
Brocard	X_3-X_6	X_{110}	X_{850}	L_{523}	X_{511}	X_{512}	X_{115}	229
Nagel	X_1-X_2	X_{190}	X_{514}	L_{649}	X_{519}	???	???	142
Bevan	X_1-X_3	X_{651}	???	L_{650}	X_{517}	X_{513}	X_{11}	128
Soddy	X_1-X_7	X_{658}	X_{3239}	X_{657}	X_{516}	X_{514}	X_{1565}	
?	X_2-X_6	X_{99}	X_{523}	L_{512}	X_{524}	X_{1499}	???	97
antiorthic	X_{44}-X_{513}	X_1	X_{75}	L_1	X_{513}	X_{517}	X_{1512}	92
?	X_1-X_6	X_{100}	X_{693}	L_{513}	X_{518}	X_{3309}	???	92
	X_1-X_4	X_{653}	???	L_{652}	X_{515}	X_{522}	???	64
	X_4-X_6	X_{107}	X_{3265}	L_{520}	X_{1503}	X_{525}	X_{1562}	40
Lemoine	X_{187}-X_{237}	X_6	X_{76}	L_2	X_{512}	X_{511}	X_{1513}	37
orthic	X_{230}-X_{231}	X_4	X_{69}	L_3	X_{523}	X_{30}	X_{1514}	29
Longchamps	X_{325}-X_{523}	X_{76}	X_6	L_{32}	X_{523}	X_{30}	X_{1531}	30
	X_3-X_8	???	???	L_{3310}	X_{952}	X_{900}	X_{3259}	14
Gergonne	X_{241}-X_{514}	X_7	X_8	L_{55}	X_{514}	X_{516}	X_{1541}	21
model	X_1-X_1	X_1	X_1	X_1	X_1	X_1	X_1	

Table 1.1: Some well-known lines

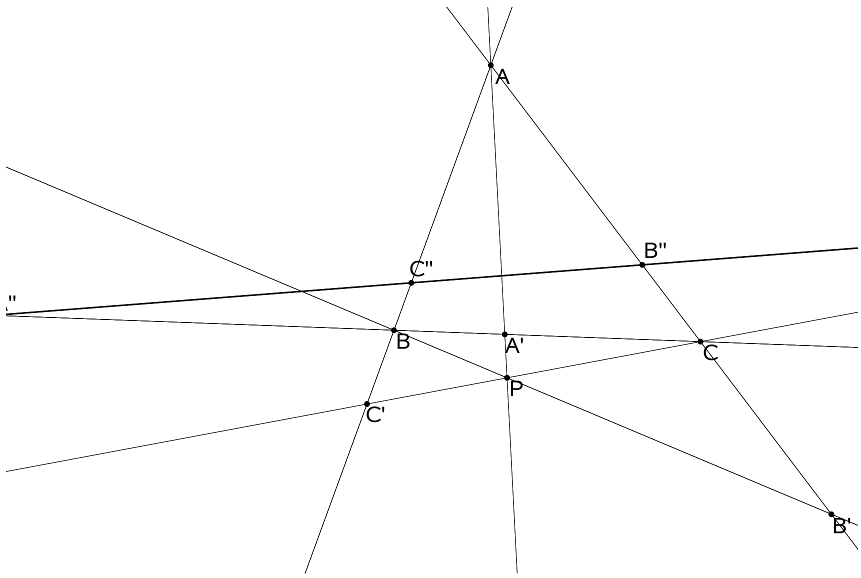


Figure 1.1: Point P and line A"B"C" are the tripolars of each other.

Proposition 1.6.14. *A symmetric parameterization in ρ, σ, τ of the points $U = u : v : w$ that lie on line $\Delta \simeq [p, q, r]$ is :*

$$U = q\tau - r\sigma : r\rho - p\tau : p\sigma - q\rho \tag{1.8}$$

A symmetric parameterization in ρ, σ, τ of the points $U = u : v : w$ that lie on line $\Delta = tripolar\,(P)$ where $P = p : q : r$ is :

$$U = p\,(\tau - \sigma) : q\,(\rho - \tau) : r\,(\sigma - \rho) \tag{1.9}$$

Proof. The first formula is $U = \Delta \wedge [\rho, \sigma, \tau]$, defining a point on a line as the (projective) intersection of this line with another one. The second one is obvious. $\square$

Proposition 1.6.15. *Line tripolar* (P) *is the locus of the tripoles* U *of the tangents to the inconic* $IC\,(P)$*. Moreover, the contact point of tripolar* (U) *is* $Q = U *_b U *_b P$.

Proof. This is the right place for the assertion, but not for its proof... Postponed to Proposition 9.5.9. □

Remark 1.6.16. ***Caveat : A triangle is not a conic***. When U lies on the polar of P wrt a conic, then P lies on the polar of U wrt the same conic. When U lies on *tripolar* (P) (the polar of P wrt triangle ABC) we have :

$$\frac{u}{p} + \frac{v}{q} + \frac{w}{r} = 0$$

and this isn't a commutative relation.

1.7 Isoconjugacy has moved

Chapter 2

Central objects

Centrality is a key notion. Emphasis on this concept was put by the founding paper of Kimberling (1998). The corresponding definitions have been tailored so that a central point is something like $I = X(1)$ or $G = X(2)$ or $O = X(3)$ etc., while a central triangle is something like ABC itself or JKL (the triangle of the excenters).

2.1 Triangle centers

Three points defines six triangles when taking the order into account. But there exists only one orthocenter. A central point is something that behaves like that. Moreover, geometrical theorems are not supposed to change when the king's foot shortens (not to speak of what happens when the king itself is shortened) : barycentric functions have to be homogeneous. This leads to the following definition.

Definition 2.1.1. A **triangle center** (or a **central line**) is a point (or a line) of the form $f(a, b, c) : f(b, c, a) : f(c, a, b)$, where f is a nonzero function satisfying two conditions:

1. f is homogeneous in a, b, c; i.e., there is a real number h such that $f(\lambda a, \lambda b, \lambda c) = \lambda^h f(a, b, c)$ for all (a, b, c) in the domain of f;

2. f is symmetric in b and c; i.e., $f(a, c, b) = f(a, b, c)$.

Definition 2.1.2. Bicentric points. Suppose $f(a, b, c) : f(b, c, a) : f(c, a, b)$ is a point that satisfies (1) in the definition of triangle center, but that $|f(a, b, c)| \neq |f(a, c, b)|$. Then $f(a, c, b) : f(b, a, c) : f(c, b, a)$ and $f(a, b, c) : f(b, c, a) : f(c, a, b)$ are bicentric points, together comprising a *bicentric pair*. Example: the Brocard points, $\omega^+ = a^2b^2 : b^2c^2 : c^2a^2$ and $\omega^- = c^2a^2 : a^2b^2 : b^2c^2$ (cf Proposition 5.7.1).

Definition 2.1.3. A **polynomial center** is a triangle center X that can be defined by X = $f(a, b, c) : f(b, c, a) : f(c, a, b)$ where f is a polynomial. Examples: X(1), X(2), $\cdots$, X(12), but not X(13).

Definition 2.1.4. A **transcendental center** is a triangle center X that cannot be defined as X = $f(a, b, c) : f(b, c, a) : f(c, a, b)$ using an algebraic function f. Examples: X(359) and X(360).

Definition 2.1.5. A **major center** is a triangle center X for which there exists a function $f(A)$ such that $X = f(A) : f(B) : f(C)$. Examples: $X(1)$, $X(2)$, $X(3)$, $X(4)$, $X(6)$. Major centers solve certain problems in functional equations (Kimberling, 1993, 1997).

Consider two examples, $X(9)$ and $X(37)$, of which first trilinears are $b + c - a$ and $b + c$, respectively. It is not clear from these trilinears that $X(9)$ is a major center, whereas $X(37)$ is not. Indeed, $X(9)$ also has first trilinear $cot(A/2)$, so that $X(9)$ is a major center, but there remains this problem: how to establish that $X(37)$ and others are not major. In April, 2008, Manol Iliev found a criterion for a triangle center to be not a major center (**reference missing**). He applied his test to the first 3236 triangle centers in ETC and found that exactly 292 of them are major, as listed Table 2.1.

1	2	3	4	6	7	8	9	13	14	15
16	17	18	19	24	25	31	32	33	34	35
36	41	47	48	49	50	55	56	57	61	62
63	68	69	75	76	77	78	79	80	85	91
92	93	94	158	173	174	179	184	186	188	200
202	203	212	215	219	220	222	236	255	258	259
264	265	266	269	273	278	279	281	289	298	299
300	301	302	303	304	305	312	317	318	319	320
323	326	328	331	340	341	345	346	348	357	358
359	360	365	366	371	372	378	393	394	400	470
471	472	473	479	480	483	485	486	491	492	506
507	508	509	554	555	556	557	558	559	560	561
562	563	571	577	601	602	603	604	605	606	607
608	728	738	847	999	1000	1028	1049	1077	1081	1082
1085	1088	1092	1093	1094	1095	1096	1102	1106	1115	1118
1119	1123	1124	1127	1128	1129	1130	1131	1132	1133	1134
1135	1136	1137	1139	1140	1143	1147	1151	1152	1250	1251
1253	1259	1260	1264	1265	1267	1270	1271	1274	1321	1322
1327	1328	1335	1336	1395	1397	1398	1399	1407	1411	1435
1442	1443	1488	1489	1496	1497	1501	1502	1583	1584	1585
1586	1593	1597	1598	1599	1600	1659	1748	1802	1804	1807
1820	1847	1857	1870	1917	1928	1969	1973	1974	1989	1993
1994	2003	2006	2052	2066	2067	2089	2151	2152	2153	2154
2160	2161	2165	2166	2174	2175	2207	2212	2289	2306	2307
2323	2351	2361	2362	2477	2671	2672	2673	2674	2962	2963
2964	2965	3043	3076	3077	3082	3083	3084	3092	3093	3179
3200	3201	3205	3206	3218	3219					

Table 2.1: Major centers

Definition 2.1.6. A **rational center** is a triangle center X that can be defined by $\mathrm{X} = f(a,b,c) : f(b,c,a) : f(c,a,b)$ where f is a polynomial that belongs to $\mathbb{C}\left(a^2,\, b^2,\, c^2,\, S\right)$ where S is the surface of ABC. When coordinates of the vertices are rational, so are the coordinates of the center.

Example 2.1.7. X(13) is not polynomial but nevertheless rational since

$$\left(\frac{1}{abc+\sqrt{3}\,(b^2+c^2-a^2)\,R} :: \right) \simeq \left(\frac{1}{4\,S+(b^2+c^2-a^2)\,\sqrt{3}} :: \right)$$

On the contrary, X(1) is polynomial, but not rational.

2.2 Central triangle

Definitions of this Section are tailored so that triangle ABC itself as well as later defined (Section 3.3) cevian and anticevian triangles are central objects. The corresponding matrices are, columnwise, as follows :

$$\mathcal{C}_P \simeq \begin{pmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{pmatrix},\ \mathcal{A}_P \simeq \begin{pmatrix} -p & p & p \\ q & -q & q \\ r & r & -r \end{pmatrix}$$

Definition 2.2.1. *A (f, g)-central triangle of type 1* is a triangle defined as :

$$[A', B', C'] \simeq \begin{pmatrix} f(a,b,c) & g(a,b,c) & g(a,b,c) \\ g(b,c,a) & f(b,c,a) & g(b,c,a) \\ g(c,a,b) & g(c,a,b) & f(c,a,b) \end{pmatrix} \tag{2.1}$$

where f, g are two center functions having the same degree of homogeneity. One of them (but not both) can be the zero function.

Example 2.2.2. Triangle ABC is $(1, 0)$ while $\mathcal{C}_P$ is $(0, f)$ and $\mathcal{A}_P$ is $(-f,\ f)$ when $P = f(a,b,c) : f(b,c,a) : f(c,a,b)$.

Definition 2.2.3. A (f, g)-*central triangle of type 2* is a triangle defined as :

$$[A', B', C'] \simeq \begin{pmatrix} f(a,b,c) & \mathbf{g}(\mathbf{a},\mathbf{c},\mathbf{b}) & g(a,b,c) \\ g(b,c,a) & f(b,c,a) & \mathbf{g}(\mathbf{b},\mathbf{a},\mathbf{c}) \\ \mathbf{g}(\mathbf{c},\mathbf{b},\mathbf{a}) & g(c,a,b) & f(c,a,b) \end{pmatrix} \tag{2.2}$$

where f, g are two center function, the second one being relaxed from condition $g(a,b,c) = g(a,c,b)$.

Example 2.2.4. Any pedal triangle that is not also a cevian triangle is a central triangle of type 2. We have :

$$f = 0,\ g\,(a,b,c) = 2\,c^2\,p\,(a,c,b) + \left(c^2 + a^2 - b^2\right)\,p\,(c,b,a)$$

2.3 Symbolic substitution

Definition 2.3.1. Symbolic substitution. Suppose $p(a,b,c)$, $q(a,b,c)$, $r(a,b,c)$ are functions of a, b, c, all of the same degree of homogeneity. As the transfigured plane consists of all functions of the form $X = x(a,b,c) : y(a,b,c) : z(a,b,c)$, the substitution indicated by

$$a \mapsto p(a,b,c), b \mapsto q(a,b,c), c \mapsto r(a,b,c)$$

maps the transfigured plane into itself.

Remark 2.3.2. Such a substitution may have no clear geometric meaning, as suggested by the name, *symbolic* substitution. On the other hand, symbolic substitutions are of geometric interest because they map lines to lines, conics to conics, cubics to cubics, and they preserve incidence.

Example 2.3.3. The symbolic substitution $(a,b,c) \mapsto (1/a, 1/b, 1/c)$ maps every triangle center to a triangle center, every pair of bicentric points to a pair of bicentric points, every circumconic to a circumconic, etc. However, when $(a,b,c) = (3,4,5)$, for example, then a, b, c are sidelengths of a Euclidean triangle, but $1/a$, $1/b$, $1/c$ are not.

Symbolic substitutions were introduced in Kimberling (2007).

Chapter 3

Cevian stuff

3.1 Centroid stuff

Definition 3.1.1. The **reflection** of point $U = u : v : w$ in point $P = p : q : r$ (not at infinity) is the point X such that :

$$X \simeq \left(\begin{array}{c} (p-q-r)u + 2p(v+w) \\ (q-p-r)v + 2q(u+w) \\ (r-p-q)w + 2r(u+v) \end{array} \right) \tag{3.1}$$

Proof. We want to obtain $X = 2P - U$ when P, U, X are finite and in normalized form. When P is finite and $U \in \mathcal{L}_\infty$ then $X = U$ (OK). Taking $P \in \mathcal{L}_\infty$ would result into $X = P$ for any value of U, not an acceptable result. □

Stratospheric proof. X is obtained under the action described by :

$$2\left(\left(\begin{array}{c} p \\ q \\ r \end{array}\right) . \mathcal{L}_\infty\right) - \left(\mathcal{L}_\infty . \left(\begin{array}{c} p \\ q \\ r \end{array}\right)\right) . \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$ □

Definition 3.1.2. **Complement** and **anticomplement** are inverse transforms defined so that the complement of a vertex of triangle ABC is the middle of the opposite side. In other words (using barycentrics),

$$\begin{array}{rcl} complement\,(U) & = & (3\,G - U)\,/2 = q + r : p + r : p + q \\ anticomplement\,(Q) & = & (3\,G - 2\,Q) = -p + q + r : p - q + r : p + q - r \end{array}$$

According to Court, p. 297, the term *complementary point* dates from 1885, and the term *anti-complementary point* dates from 1886.

Definition 3.1.3. The **medial triangle**$\mathcal{C}_2$ is the complement of triangle ABC. The A-vertex of $\mathcal{C}_2$ is the middle of segment BC, and cyclically. In other words :

$$\mathcal{C}_2 = \left(\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right)$$

Definition 3.1.4. The **antimedial triangle**$\mathcal{A}_2$ is the anticomplement of triangle ABC (and is also called the anticomplementary triangle). Each sideline of $\mathcal{A}_2$ contains a vertex of ABC and corresponding sidelines of both triangles are parallel. In other words :

$$\mathcal{A}_2 = \left(\begin{array}{ccc} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right)$$

These triangles are the model used to define the next coming cevian and anticevian triangles (they are the triangles related to the centroid X_2).

3.2 Harmonic division

Proposition 3.2.1. Cross ratio of four points. *Using barycentrics, and assuming that points $P_j = p_j : q_j : r_j$ are aligned then, most of the time, their cross-ratio is given by :*

$$harm\left(P_1,\, P_2,\, P_3,\, P_4\right) = \frac{\left(s_4\, p_2 - s_2 p_4\right)\left(s_3\, p_1 - s_1\, p_3\right)}{\left(s_4\, p_1 - s_1\, p_4\right)\left(s_3\, p_2 - s_2\, p_3\right)} \tag{3.2}$$

where $s_j = p_j + q_j + r_j$.

Proof. We assume that $\#\left\{P_j\right\} \geq 3$. When two of the p_j are 0, all the four are 0, the formula is indetermined, and another coordinate has to be chosen. Otherwise, this is only $\left(\left(z_4 - z_2\right) / \left(z_4 - z_1\right)\right) \div \left(\left(z_3 - z_2\right) / \left(z_3 - z_1\right)\right)$ applied to the normalized barycentrics. □

Definition 3.2.2. Four points A, B, J, K are in **harmonic division** when :

$$\frac{\overline{AJ}}{\overline{BJ}} \div \frac{\overline{AK}}{\overline{BK}} = -1$$

Since cross-ratio is a projective invariant, this relationship is carried by perspective.

Proposition 3.2.3. *The fourth of an harmonic division (later shortened into* **fourth harmonic***) can be constructed using a point out of the line of the first three. In Figure 3.1, points A, P, C and external D are given. Then Δ is the parallel to AD through C, E is $DP \cap \Delta$ and F the reflection of E into C. Point Q result from the fact that division E, F, C, ∞_Δ is clearly harmonic.*

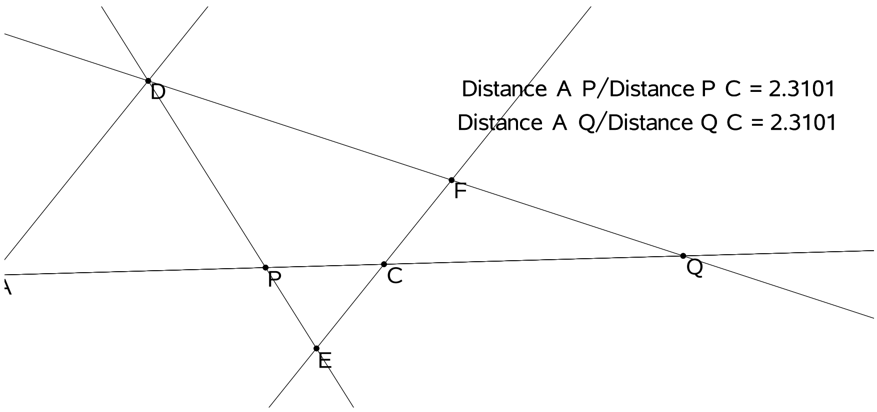


Figure 3.1: Obtain Q from A,P,C and auxiliary D

Remark 3.2.4. In the cartesian plane, the fourth harmonic is also the reflection of the third point into the circle having the first two as diameter. This will be restated in a projective way in Subsection 14.1

3.3 Cevian, anticevian, cocevian triangles

Theorem 3.3.1 (Ceva)**.** *Let $A' \in BC$, $B' \in CA$ and $C' \in AB$ be three points on the sidelines of triangle ABC, but different from the vertices. Then lines AA', BB', CC' are concurrent if and only if :*

$$\frac{\overline{AB'}}{\overline{CB'}} \frac{\overline{BC'}}{\overline{AC'}} \frac{\overline{CA'}}{\overline{BA'}} = -1$$

Proof. The usual proof uses Menelaus theorem. Another proof, using determinants, is given below. □

Definition 3.3.2. Cevian triangle. Let P be a point not on a sideline of ABC. The lines AP, BP, CP are the *cevians* of P. Let $A_p = AP \cap BC$. Define B_p and C_p cyclically. Triangle $A_pB_pC_p$ is the cevian triangle of triangle ABC.

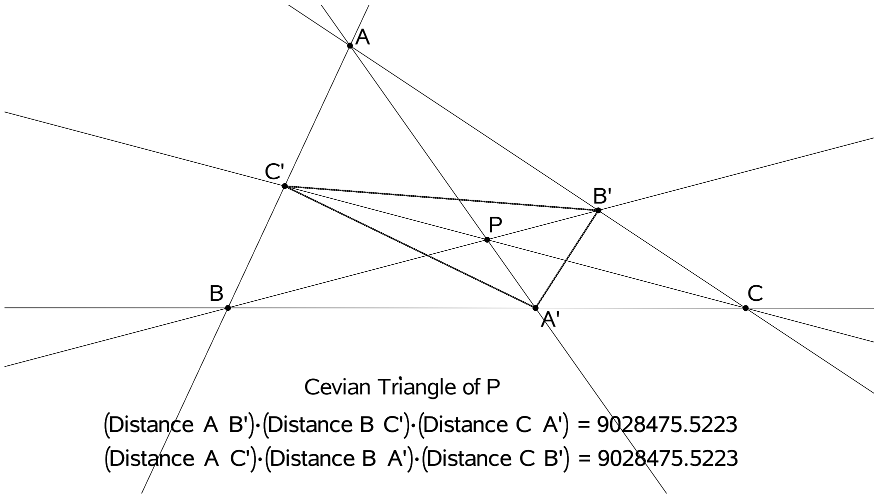


Figure 3.2: Ceva's Theorem

$$P \simeq \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Longrightarrow cevian(P) = \begin{pmatrix} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{pmatrix} \tag{3.3}$$

Example 3.3.3. Examples of cevian triangles are given in Table 3.1.

cevian				bary	G	O	I
incentral	17.2.3	incenter	X(1)	a			
medial	17.2.1	centroid	X(2)	1	X(2)	X(5)	X(10)
orthic	17.2.4	H-center	X(4)	$\tan A$	X(51)	X(5)	X(4)
intouch	17.2.5	Gergonne	X(7)	$(b+c-a)^{-1}$			
extouch	17.2.7	Nagel	X(8)	$b+c-a$	X(210)	X(1158)	
anticevian				bary	G	O	I
excentral	17.2.8	incenter	X(1)	a	X(165)	X(40)	X(164)
antimedial	17.2.2	centroid	X(2)	1	X(2)	X(4)	X(8)
tangential	17.2.9	Lemoine	X(6)	a^2	X(154)	X(26)	X(3)
other							
Fuhrmann	Subsection 17.2.10						
star	Subsection 17.2.11				X(3817)	X(946)	?

Table 3.1: Some well-known triangles

Proposition 3.3.4. *Equation 3.4 gives the condition for an inscribed triangle to be the Cevian triangle of some point P.*

$$\begin{pmatrix} 0 & p_2 & p_3 \\ q_1 & 0 & q_3 \\ r_1 & r_2 & 0 \end{pmatrix} \text{ is Cevian } \Longleftrightarrow p_2\, q_3\, r_1 - p_3\, q_1\, r_2 = 0 \tag{3.4}$$

$$\text{and then } p : q : r = r_1 p_2 : q_1 r_2 : r_1 r_2$$

An absolutely hopeless formula, but nevertheless more "obviously symmetrical" is :

$$p : q : r = \sqrt[3]{p_2^2\, p_3^2\, q_1\, r_1} \; : \; \sqrt[3]{q_3^2\, q_1^2\, p_2\, r_2} \; : \; \sqrt[3]{r_1^2\, r_2^2\, p_3\, q_3}$$

Proof. Line AP_1 is $(0,\, r_1,\, -q_1)$ and cyclically. These lines are concurrent when their determinant vanishes. Their common point is then given by any column of the adjoint matrix, or the cubic root of their product. □

Proposition 3.3.5. Isotomic conjugacy. *Suppose $U = u : v : w$ is a point not on a sideline of ABC. Let $A_U = AU \cap BC$ and define B_U, C_U cyclically. Reflect $A_U B_U C_U$ about the midpoints of sides BC, CA, AB, respectively, to obtain points A', B', C'. Then lines AA', BB', CC' are concurrent. Their common intersection is called the isotomic conjugate of U. The corresponding barycentrics are :*

$$isot\,(u : v : w) = \frac{1}{u} : \frac{1}{v} : \frac{1}{w} \tag{3.5}$$

Proof. Immediate computation. □

Remark 3.3.6. The fixed points of this transform are the gravity center and its relatives, so that $isot\,(U) = U_G^{\#}$ while $U *_b isot\,(U) =$X(2). It should be noticed that $X\,(2)$ plays together the role of points F and P of Definition 1.5.10.

Definition 3.3.7. Anticevian triangle. Let P be a point not on a sideline of ABC. Let $A_p = AP \cap BC$. (The lines AP, BP, CP are the *cevians* of P, and triangle $A_pB_pC_p$ is the *cevian triangle* of P). Let P_A be the harmonic conjugate of P with respect to A and A_p. Define P_B and P_C cyclically. Triangle $P_AP_BP_C$ is the anticevian triangle of triangle ABC.

$$P \simeq \begin{pmatrix} p \\ q \\ r \end{pmatrix} \implies anticevian(P) = \begin{pmatrix} -p & p & p \\ q & -q & q \\ r & r & -r \end{pmatrix} \tag{3.6}$$

Example 3.3.8. Examples of anticevian triangles are given in Table 3.1.

Proposition 3.3.9. *(1) Triangle ABC is inscribed in triangle $P_AP_BP_C$. (2) ABC is the cevian triangle of P wrt the anticevian triangle. (3) Anticevian triangle of point P_A wrt ABC is PP_CP_B (the two remaining points are permutated).*

3.4 Transversal lines, Menelaus and Miquel theorems

Proposition 3.4.1 (Genuine Menelaus theorem)**.** *Let $A' \in BC$, $B' \in CA$ and $C' \in AB$ be three points on the sidelines of triangle ABC, but different from the vertices. Then A', B', C' are collinear if and only if :*

$$\frac{\overline{AB'}}{\overline{CB'}}\,\frac{\overline{BC'}}{\overline{AC'}}\,\frac{\overline{CA'}}{\overline{BA'}} = +1$$

Proof. Let us parameterize the situation by $A' = k_aB + (1-k_a)C$, etc. Alignment is described by :

$$\begin{vmatrix} 0 & 1-k_b & k_c \\ k_a & 0 & 1-k_c \\ 1-k_a & k_b & 0 \end{vmatrix} = 0$$

while $A' - B = (1-k_a)\,(C-B)$ and $A' - C = k_a\,(B-C)$. □

Proposition 3.4.2. ***Cocevian triangle***. *Let $P = p : q : r$ be a point not on a sideline of ABC, and let $A_PB_PC_P$, $P_AP_BP_C$ be respectively the cevian and anticevian vertices of P. Let T_A be the harmonic conjugate of A_P wrt B and C. This is also the harmonic conjugate of A wrt P_B and P_C. Define T_B and T_C cyclically. The cocevian triangle of P is the triangle $T_A\,T_B\,T_C$. This triangle is degenerate and its vertices belong to the **tripolar line** of P : $x/p + y/q + z/r = 0$.*

$$P \simeq \begin{pmatrix} p \\ q \\ r \end{pmatrix} \implies cocevian(P) = \begin{pmatrix} 0 & p & -p \\ -q & 0 & q \\ r & -r & 0 \end{pmatrix} \tag{3.7}$$

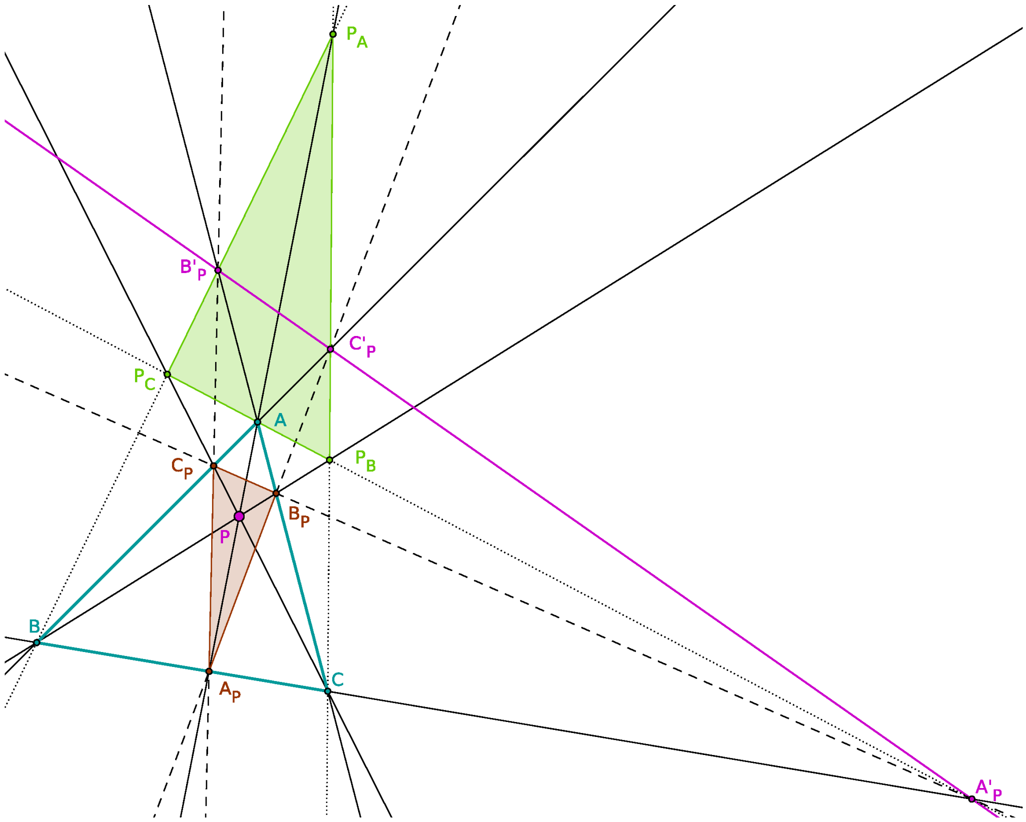


Figure 3.3: Cevian, anticevian and cocevian triangles

Proof. We have $A_P \simeq 0 : q : r$ so that $T_P \doteq 0 : -q : r$ is the fourth harmonic of B, C, A_P. We have $P_B \simeq p : -q : r$, $P_C \simeq p : q : -r$ so that $P_B + P_C \simeq A$ while $P_B - P_C \simeq T_A$ and $(P_B, P_C, A, T_A) = -1$ is proved. Alignment is obvious from the determinant. Menelaus condition is $pqr - prq = 0$. Relation with the tripolar, i.e. the "transpose and reciprocate" line is clear. □

Proposition 3.4.3 (Miguel theorem)**.** . *Let $A' \in BC$, $B' \in CA$ and $C' \in AB$ be three points on the sidelines of triangle ABC, but different from the vertices. Then circles $AB'C'$, $A'BC'$, $A'B'C$ are passing through a same point M, the Miquel point of $A'B'C'$ wrt ABC.*

Proof. Equation of circle $AB'C'$ is :

$$a^2 y\, z + b^2 z\, x + c^2 x\, y - (x + y + z)\left(y\, k_c\, c^2 + z\,(1 - k_b)\, b^2\right)$$

Therefore their last common point is :

$$M \simeq \left(\begin{array}{c} a^2\left(k_a\,(k_a - 1)\, a^2 + (k_a - 1)\,(k_b - 1)\, b^2 + c^2 k_c k_a\right) \\ b^2\left(k_a k_b\, a^2 + k_b\,(k_b - 1)\; b^2 + (k_b - 1)\,(k_c - 1)\, c^2\right) \\ c^2\left((k_c - 1)\,(k_a - 1)\, a^2 + b^2 k_b k_c + k_c\,(k_c - 1)\, c^2\right) \end{array}\right)$$

Since this expression is symmetric, this expression is on the third circle. □

Theorem 3.4.4 (Extended Menelaus theorem)**.** *Let $A' \in BC$, $B' \in CA$ and $C' \in AB$ be three points on the sidelines of triangle ABC, but different from the vertices. All the following are necessary and sufficient conditions for A', B', C' to be collinear :*
(i) the Menelaus condition : $((1 - k_a)\,(1 - k_b)\,(1 - k_c) + k_a k_b k_c) = 0$
(ii) the midpoints $M_a = (A + A')/2$, $M_b = (B + B')/2$, $M_c = (C + C')/2$ are on the same line (the so-called Newton line of the quadrilateral)
(iii) the Miquel point of $A'B'C'$ is on the circumcircle of ABC.
(iv) the homographic application φ defined in $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$ by $A \mapsto A'$, $B \mapsto B'$, $C \mapsto C'$ is involutory.

Proof. (i) obvious ; (ii) determinant ; (iii) condition for $M \in \Gamma$ is the Menelaus condition times an uggly factor that can be written as :

$$\left(2\,a^2k_a + 2\,b^2k_b + 2\,c^2k_c - c^2 - b^2 - a^2\right)^2 + 16\,S^2$$

(iv) Condition for φ to be involutory is $\det_3[1,\, \alpha+\alpha',\, \alpha\,\alpha'] = 0$. We obtain Vdm times the Menelaus condition (see Section 12.3 for notations and more details). □

Proposition 3.4.5. *When P is neither a vertex nor $X(2)$, the centroid of the cocevian triangle is well defined (perhaps on $\mathcal{L}_\infty$), and called the tripolar centroid of P (Stothers, 2003b).*

$$TG\,(P) = p(q-r)(2p-q-r) : q(p-r)(p-2q+r) : r(q-p)(p+q-2r)$$

Remark 3.4.6. When $q = r$ then $TG\,(P) = 0 : 1 : -1$ (at infinity on BC). When all p, q, r are different, $TG\,(P)$ is a finite point.

Proposition 3.4.7. *When all p, q, r are different, then it exists exactly another point that shares the same $TG\,(P)$, namely :*

$$other\,(P) = \frac{q+r-2\,p}{rp+qp-2\,qr} : \frac{r+p-2\,q}{qp+qr-2\,rp} : \frac{p+q-2\,r}{qr+rp-2\,qp}$$

Proof. Direct computation. When eliminating k, u, v in $TG\,(P) = k\,TG\,(U)$, special cases are p, $q-r$, u, $v-w$, $qw-rv$, $2p-q-r$ et cyclically. For all named points X, it happens that $other\,(X)$ is not named. □

Example 3.4.8. Points $X(1635)$ to $X(1651)$ are defined that way. Examples include :

1	1635	98	1640	263	2491	525	1650
3	1636	99	1641	512	1645	648	1651
4	1637	100	1642	513	1646	957	3310
6	351	105	1643	514	1647	1002	665
7	1638	190	1644	523	1648	1022	244
8	1639	262	3569	524	1649	2394	125

3.5 Perspectivity

Definition 3.5.1. Vertex trigon, vertex triangle. Let $\mathcal{T}_1 = A_1B_1C_1$ and $\mathcal{T}_2 = A_2B_2C_2$ be two triangles. Their vertex trigon is the set of three lines $L_a = A_1A_2$, $L_b = B_1B_2$, $L_c = C_1C_2$, while their vertex triangle is the set of points $A_3 = L_b \cap L_c$, etc.

Remark 3.5.2. The generic rank of the matrix of the vertex trigon is 3. An interresting case is $rank = 2$ while $rank = 1$ is very weird (all points on the same line). On the contrary, since the vertex triangle matrix is the adjoint of the trigon matrix, the vertex triangle is either generic ($rank = 3$) or reduced to a point ($rank = 1$) .

Definition 3.5.3. Perspector. Let $\mathcal{T}_1 = A_1B_1C_1$ and $\mathcal{T}_2 = A_2B_2C_2$ be two triangles. When the vertex trigon, i.e. the lines A_1A_2, B_1B_2, C_1C_2, concur at some point P, this point is called the *perspector* (replacing *center of perspective*) of the (ordered) triangles.

Definition 3.5.4. Homoltriangle. The homoltriangle of two given triangles $\mathcal{T}_1 = A_1B_1C_1$ and $\mathcal{T}_2 = A_2B_2C_2$ is defined as the triangle $\mathcal{T}_3 = B_1C_1 \cap B_2C_2,\ C_1A_1 \cap C_2A_2,\ A_1B_1 \cap A_2B_2$. Its name comes from the fact we are intersecting the homologue sidelines.

Definition 3.5.5. Crosstriangle. The crosstriangle of two given triangles $\mathcal{T}_1 = A_1B_1C_1$ and $\mathcal{T}_2 = A_2B_2C_2$ is defined as the triangle $\mathcal{T}_3 = B_1C_2 \cap B_2C_1,\ C_1A_2 \cap C_2A_1,\ A_1B_2 \cap A_2B_1$. Its name comes from the fact we are crossing the vertices of the homologue sidelines.

Proposition 3.5.6. Perspectrix. *When two triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ are perspective to each other, their crosstriangle $\mathcal{T}_3 = B_1C_2 \cap B_2C_1,\ C_1A_2 \cap C_2A_1,\ A_1B_2 \cap A_2B_1$ degenerates to a line, called the perspectrix (replacing* axis of perspective*) of the triangles.*

Proof. This is an application of Desargues theorem. Another proof is :

$$\det \mathcal{T}_3 = \det \mathcal{T}_1 \, \det \mathcal{T}_2 \, \det \left(A_1 A_2, \, B_1 B_2, \, C_1 C_2 \right)$$ □

Example 3.5.7. Let $\mathcal{T}_1$ be the reference triangle ABC and $\mathcal{T}_2$ the cevian triangle $A_p B_p C_p$ of a point P. Then (i) P is the perspector of both triangles, (ii) the Desargues' points are the vertices of the cocevian (degenerate) triangle and (iii) the perspectrix is the trilinear polar of P. Let us recall that :

$$\begin{gathered} A_p = 0 : q : r, \; B_p = p : 0 : r, \; C_p = p : q : 0 \\ A' = 0 : -q : r, \quad B' = p : 0 : -r, \; C' = -p : q : 0 \\ (x : y : z) \in A'B'C' \; is \; q\,r\,x + r\,p\,y + p\,q\,z = 0 \end{gathered}$$

Example 3.5.8. Reference triangle and the anticevian triangle $P_A P_B P_C$ of a point P is another example. Let us recall that :

$$P_A = -p : q : r, \; P_B = p : -q : r, \; P_C = p : q : -r$$

Proposition 3.5.9. Perspectivity with ABC. *(1) Let $\mathcal{T}_1 = ABC$ and $\mathcal{T}_2 = UVW$ be two triangles that admits P as common perspector. Use barycentrics with respect to ABC and note $P = p : q : r$. Then constants u, v, w can be found that allow to write UVW in the following "synchronized" form :*

$$\mathcal{T}_2 = \left(\begin{array}{ccc} u & p & p \\ q & v & q \\ r & r & w \end{array} \right)$$

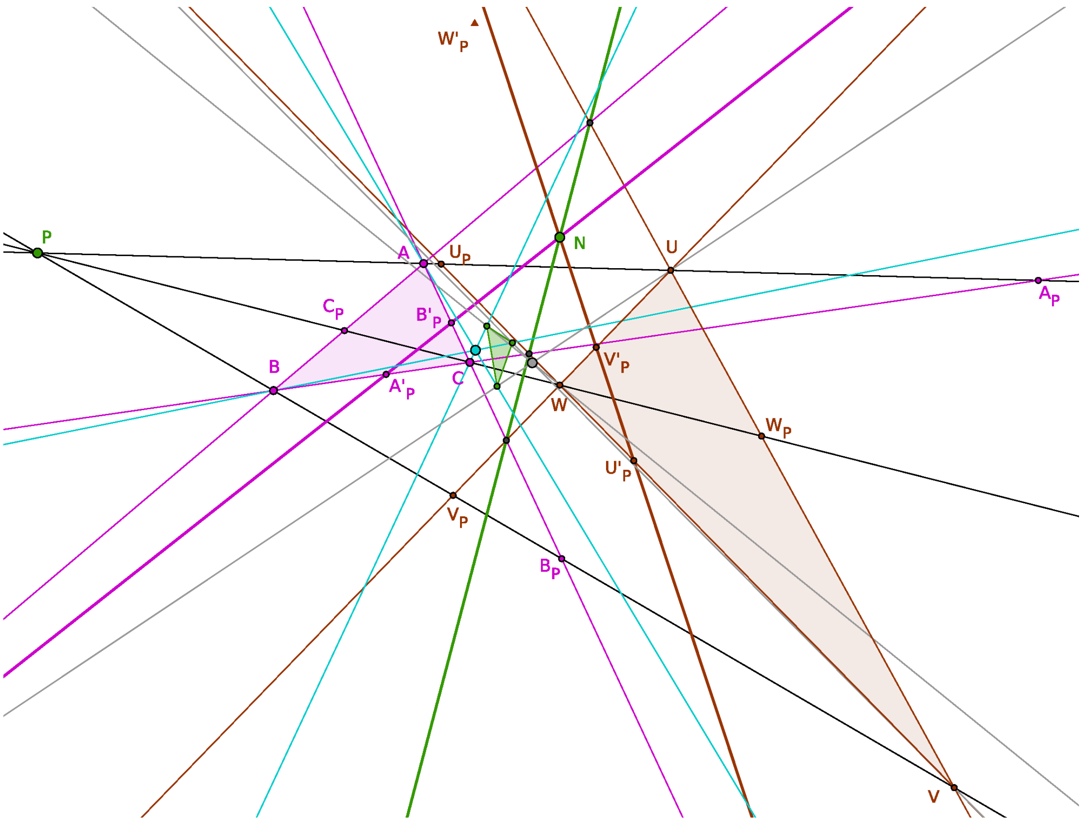


Figure 3.4: Triangles ABC and UVW that admit P as perspector.

(2) Suppose that perspector P is not on a sideline of ABC. Note $\mathcal{T}_{1P}$ and $\mathcal{T}_{2P}$ the cevian triangles of P wrt $\mathcal{T}_1$ and $\mathcal{T}_2$. The four triangles share the same perspector $P = p : q : r$, while

their perspectrices are :

$$\begin{array}{rll} \mathcal{T}_1 & \mathcal{T}_2 & \Delta_1 \doteq \left[\begin{array}{ccc} (q-v)\,(r-w) & (r-w)\,(p-u) & (p-u)\,(q-v) \end{array}\right] \\ \mathcal{T}_1 & \mathcal{T}_{1P} & \Delta_P = \left[\begin{array}{ccc} qr & pr & pq \end{array}\right] \\ \mathcal{T}_1 & \mathcal{T}_{2P} & pqr\,\Delta_1 - \det \mathcal{T}_2\,\Delta_P \\ \mathcal{T}_2 & \mathcal{T}_{1P} & 2\,pqr\,\Delta_1 - \det \mathcal{T}_2\,\Delta_P \\ \mathcal{T}_2 & \mathcal{T}_{2P} & 3\,pqr\,\Delta_1 - \det \mathcal{T}_2\,\Delta_P \\ \mathcal{T}_{1P} & \mathcal{T}_{2P} & \Delta_1 \end{array}$$

and therefore are concurrent. This occurs at :

$$N \simeq \left(\begin{array}{c} (qw-rv)\,(p-u)\,p \\ (ru-pw)\,(q-v)\,q \\ (pv-qu)\,(r-w)\,r \end{array}\right)$$

(3) Suppose aditionally that $\mathcal{T}_2$ is not the anticevian of P wrt ABC. Let be $A_3 = BW \cap CV$ and cyclically. Then triangle $\mathcal{T}_3 = A_3B_3C_3$ is :

$$\mathcal{T}_3 \doteq cross\,(\mathcal{T}_1,\,\mathcal{T}_2) = \left(\begin{array}{ccc} p & u & u \\ v & q & v \\ w & w & r \end{array}\right)$$

Triangles $\mathcal{T}_1$ and $\mathcal{T}_3$ admit $Q = u : v : w$ as perspector, while triangles $\mathcal{T}_2$and $\mathcal{T}_3$ admit $R = p+u : q+v : r+w$ as perspector.

Proof. (1) A_1 is colinear with A and $AP \cap BC$. (2) These results are obtained with a factor pqr, so that P on a sideline must be discarded. (3) Thereafter, point R is obtained, for example, with factor $pq\,(w-r)\,(uv-pq)$. If all $w-r$ vanish then $\mathcal{T}_2$ is totally degenerate. If all $uv-pq$ vanish then $u = \pm p$ etc (same signs), discarding also the anticevian triangle (in this case $\mathcal{T}_3 = \mathcal{T}_2$, and R is indeterminate). □

3.6 Cevian nests

Definition 3.6.1. Cevian nest. Suppose $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are triangles and that $\mathcal{T}_1$ is inscribed in $\mathcal{T}_2$ and $\mathcal{T}_2$ is inscribed in $\mathcal{T}_3$. If any two of the tree triangles are perspective, it is well known that each is perspective to the third : $\mathcal{T}_1$ is a cevian triangle of $\mathcal{T}_2$ for some point P, $\mathcal{T}_3$ is an anticevian triangle of $\mathcal{T}_2$ for some point U and $\mathcal{T}_1$, $\mathcal{T}_3$ are perspective wrt some point X. Such configuration is called a cevian nest.

Proposition 3.6.2. *Map $P = P\,(U, X)$ giving the perspector of $\mathcal{T}_1$, $\mathcal{T}_2$ from the other two perspectors of a cevian nest is symmetric, while –for a given P– map $X = X\,(U)$ is involutory.*

Proof. Given the vertices S_i $(i = 1, 2, 3)$ of triangle $\mathcal{T}_3$ and perspector U, the vertices S_i $(i = 4, 5, 6)$ of triangle $\mathcal{T}_2$ are obtained by $S_4 = (S_1 \wedge U) \wedge (S_2 \wedge S_3)$ and cyclically. Process can be iterated, obtaining vertices S_i $(i = 7, 8, 9)$ of $\mathcal{T}_1$. Then X is obtained as $X = (S_1 \wedge S_7) \wedge (S_2 \wedge S_8)$. It can be checked that substituting U by X (and keeping everything else unchanged) leads back to U, proving the second part. The first part follows immediately. □

When triangle ABC belongs to such a nest, three possibilities can occur. The corresponding operations are summarized in Table 3.2, where "mul" stands for multiplication (giving P) and "div" for the converse operation. The Kimberling's name is also given.

3.7 The cross case (aka case I, cev of cev)

Definition 3.7.1. Crossmul(U,X). As in Table 3.2 (I), let $\mathcal{T}_3$ (the biggest triangle) be ABC, $\mathcal{T}_2 = A_U B_U C_U$ the (usual) cevian triangle of U and $\mathcal{T}_1 = A'B'C'$ the triangle inscribed in $\mathcal{T}_2$ obtained by $A' = AX \cap B_U C_U$ and cyclically. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ have a perspector (P), and mapping $(U, X) \mapsto P$ is called cross-multiplication.

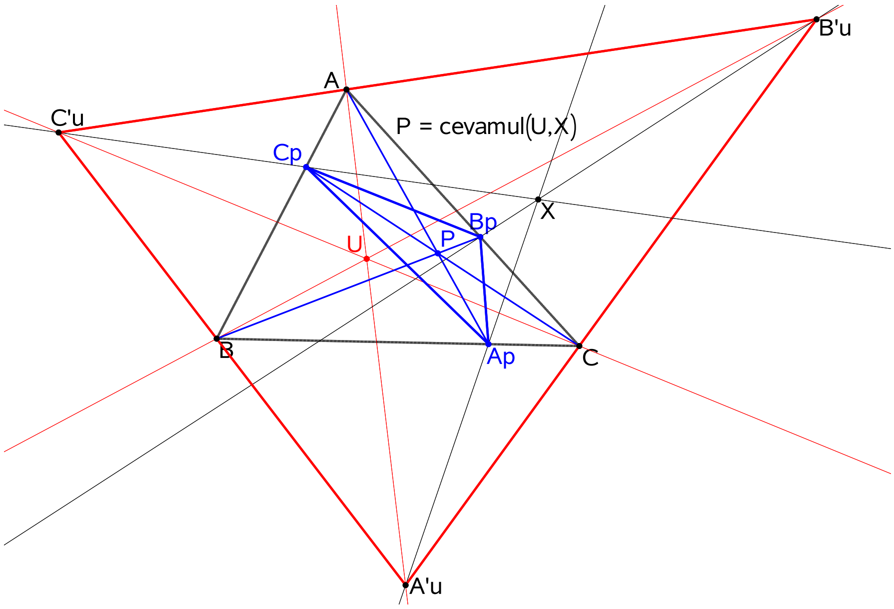


Figure 3.5: P is cevamul(U,X)

$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	X, P	Kimberling
$\mathcal{C}_P$ wrt $\mathcal{T}_2$	$\mathcal{C}_U$	ABC	$X = crossdiv\,(P,U)$ $P = crossmul\,(U,X)$	cross − conj cross − point
$\mathcal{C}_P$	ABC	$\mathcal{A}_U$	$X = cevadiv\,(P,U)$ $P = cevamul\,(U,X)$	ceva − conj ceva − point
ABC	$\mathcal{A}_P$	$\mathcal{A}_U$ wrt $\mathcal{T}_2$	$X = sqrtdiv\,(P,U)$ $P = sqrtmul\,(U,X)$	

All these operations are (globally) type-keeping, since they transform points into constructible points.

Table 3.2: Three cases of cevian nets

Definition 3.7.2. Crossdiv(P,U). As in Table 3.2 (I), let $\mathcal{T}_3$ (the biggest triangle) be ABC, $\mathcal{T}_2 = A_U B_U C_U$ the (usual) cevian triangle of U and $\mathcal{T}_1 = A'B'C'$ the cevian triangle of P wrt $\mathcal{T}_2$, obtained by $A' = A_U P \cap B_U C_U$ and cyclically. Then $\mathcal{T}_1$ and $\mathcal{T}_3$ have a perspector (X), and mapping $(P,U) \mapsto X$ is called cross-division.

Proposition 3.7.3. *Computing rules of crossmul and crossdiv are given (using barycentrics) in Figure 3.6. Map $(U,X) \mapsto P$ is commutative. That's a reason to call it "cross multiplication" instead of the Kimberling's crosspoint. Map $(P,U) \mapsto X$ behaves wrt crossmul like division behaves wrt ordinary multiplication. That's a reason to call it "cross division" instead of "cross conjugacy". These points are introduced in Kimberling (1998) with the notation $X = C(P,U)$ and also $X = PcU$.*

Proof. Barycentrics $p : q : r$ are defining point P with respect to triangle $\mathcal{T}_3$. Call $\overline{P} : Q : R$ its barycentrics with respect to triangle $\mathcal{T}_2$, so that $(p : q : r) = \boxed{\mathcal{T}_2}\left(\overline{P} : Q : R\right)$. Then :

$$\mathcal{T}_1 \simeq \begin{pmatrix} 0 & u & u \\ v & 0 & v \\ w & w & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \overline{P} & \overline{P} \\ Q & 0 & Q \\ R & R & 0 \end{pmatrix} \cdot \begin{pmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{pmatrix} \simeq \begin{pmatrix} \dfrac{u\,(Q+R)}{QR} & \dfrac{u}{\overline{P}} & \dfrac{u}{\overline{P}} \\ \dfrac{v}{Q} & \dfrac{v\,(\overline{P}+R)}{\overline{P}R} & \dfrac{v}{Q} \\ \dfrac{w}{R} & \dfrac{w}{R} & \dfrac{w\,(\overline{P}+Q)}{\overline{P}Q} \end{pmatrix}$$

where the diagonal matrix had been chosen to "synchronize" the columns of triangle $\mathcal{T}_2$. Then Proposition 3.5.9 shows that $\mathcal{T}_3$ and $\mathcal{T}_1$ are perspective wrt point $u/\overline{P} : v/Q : w/R$. □

$$\begin{tikzcd}
 & P/U \arrow[dl, "*_b U"'] & & X/U \arrow[ll, "\text{complem} \circ \text{isotom}"'] & \\
P & & & & (U,X) \arrow[ul, "*_b U^{-1}"'] \arrow[llll, dotted, "\text{crossmul}"] \arrow[dl, "*_b X^{-1}"] \\
 & U/X \arrow[ul, "*_b X"] & & P/X \arrow[ll, "\text{complem} \circ \text{isotom}"] &
\end{tikzcd}$$

$$crossmul\,(u:v:w,\ x:y:z) = (v\,z + w\,y)\,u\,x : (u\,z + w\,x)\,v\,y : (u\,y + v\,x)\,w\,z \tag{3.8}$$

$$\begin{tikzcd}
X & X/U \arrow[l, "*_b U"'] & & P/U \arrow[ll, "\text{isotom} \circ \text{anticomplem}"'] & (P,U) \arrow[l, "*_b U^{-1}"'] \arrow[llll, dotted, bend left, "\text{crossdiv}"]
\end{tikzcd}$$

$$crossdiv\,(p:q:r,\ u:v:w) = \frac{u}{quw + ruv - pvw} : \frac{v}{pvw + ruv - quw} : \frac{w}{pvw + quw - ruv} \tag{3.9}$$

Figure 3.6: crossmul, crossdiv

Remark 3.7.4. The factorisation given can be interpreted in terms of isoconjugacies (see Chapter 14). First compute what happens when $U =$X(2) and obtain complem $\circ$ isotom. Then transmute this map by the transform $*_b U$ that fixes A, B, C and send $G =$X(2) onto U.

Proposition 3.7.5. *The crossmul P of U, X is also the point of concurrence of (1) the line through points $AX \cap BU$ and $AU \cap BX$, (2) the line through points $BX \cap CU$ and $BU \cap CX$, (3) the line through points $CX \cap AU$ and $CU \cap AX$.*

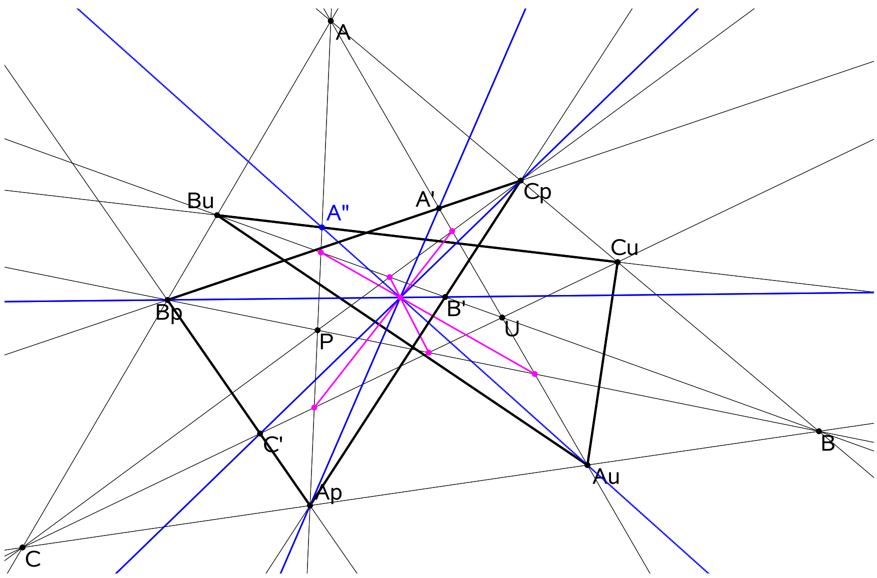


Figure 3.7: P is crossmul of U and X

3.8 The ceva case (aka case II, cev and acev)

Definition 3.8.1. Cevamul(U,X). As in Table 3.2 (II), let $\mathcal{T}_2$ (the middle triangle) be ABC, $\mathcal{T}_3 = U_A U_B U_C$ the anticevian triangle of U and $\mathcal{T}_1 = A'B'C'$ be the triangle inscribed in $\mathcal{T}_2 = ABC$

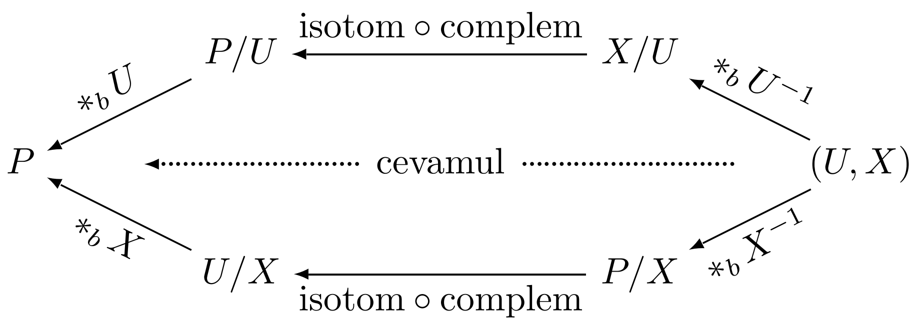
obtained by $A' = U_A X \cap BC$ and cyclically. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ have a perspector (i.e. $\mathcal{T}_1$ is the cevian triangle of some point P), and mapping $(U, X) \mapsto P$ is called ceva-multiplication.

Definition 3.8.2. Cevadiv(P,U). As in Table 3.2 (II), let $\mathcal{T}_2$ (the middle triangle) be ABC, $\mathcal{T}_3 = U_A U_B U_C$ the anticevian triangle of U and $\mathcal{T}_1 = A_P B_P C_P$ be the cevian triangle of P, obtained (as usual) by $A_P = AP \cap BC$ and cyclically. Then $\mathcal{T}_1$ and $\mathcal{T}_3$ have a perspector (X), and mapping $(P, U) \mapsto X$ is called ceva-division.

Proposition 3.8.3. *Computing rules of cevamul and cevadiv are given (using barycentrics) in Figure 3.8. Map $(U, X) \mapsto P$ is commutative i.e. :*

$$\begin{cases} X\, is\, the\, perspector\, of\, \mathcal{C}_P\, and\, \mathcal{A}_U \\ U\, is\, the\, perspector\, of\, \mathcal{C}_P\, and\, \mathcal{A}_X \end{cases}$$

That's a reason to call this map "ceva multiplication" instead of the Kimberling's cevapoint. Map $(P, U) \mapsto X$ behaves wrt cevamul like division behaves wrt ordinary multiplication. That's a reason to call it "ceva division" instead of "ceva conjugacy". These points are introduced in Kimberling (1998) with the notation $X = P \copyright U$.



$$\begin{aligned} &cevamul\,(u : v : w,\; x : y : z) = \\ &\qquad (uz + wx)(uy + vx) : (vz + wy)\,(uy + vx) : (vz + wy)\,(uz + wx) \end{aligned} \tag{3.10}$$

$$\begin{aligned} &cevadiv\,(p : q : r,\; u : v : w) = \\ &\qquad u(-q\,r\,u + r\,p\,v + p\,q\,w) : v(q\,r\,u - r\,p\,v + p\,q\,w) : w(q\,r\,u + r\,p\,v - p\,q\,w) \end{aligned} \tag{3.11}$$

Figure 3.8: cevamul, cevadiv

Proposition 3.8.4. *Suppose $U = u : v : w$ and $X = x : y : z$ are distinct points, neither lying on a sideline of ABC. Let $\mathcal{A}_x = X_A X_B X_C$ and $\mathcal{A}_u = U_A U_B U_C$ be the anticevian triangles of X and U (wrt ABC). Define A' as $U_A X \cap X_A U$ and B', C' cyclically (Figure 3.5). Then triangle $A'B'C'$ is inscribed in ABC and is, in fact, the cevian triangle of $P = cevamul\,(U, V)$.*

Proof. Direct computation. □

Proposition 3.8.5 (Floor Van Lamoen (2003/10/17))**.** *Point cevamul(U, X) can be constructed from the cevian triangles : let $A_U B_U C_U$ be the cevian triangle of U, and $A_X B_X C_X$ the cevian triangle of X. Define :*

$A_{ux} = A_u C_x \cap A_x B_u$	$A_{xu} = A_u B_x \cap A_x C_u$
$B_{ux} = B_u A_x \cap B_x C_u$	$B_{xu} = B_u C_x \cap B_x A_u$
$C_{ux} = C_u B_x \cap C_x A_u$	$C_{xu} = C_u A_x \cap C_x B_u$

Then, as seen in Figure 3.9, triangle ABC is perspective to both triangles A_{ux}, B_{ux}, C_{ux} and A_{xu}, B_{xu}, C_{xu}, and the perspector in both cases is the cevamul (P, U).

Proof. Direct computation. □

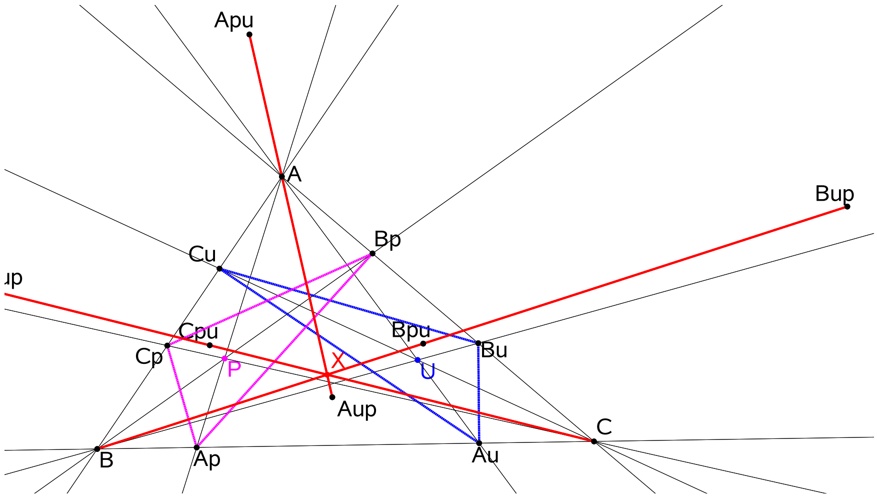


Figure 3.9: Lamoen's construction of X=cevamul(P,U)

3.9 The square case (aka case III, acev of acev)

Definition 3.9.1. sqrtdiv(P,U). As in Table 3.2 (III), let $\mathcal{T}_1$ (the smallest triangle) be ABC, $\mathcal{T}_2 = P_A P_B P_C$ the anticevian triangle of P and $\mathcal{T}_3$ be the anticevian triangle of U wrt $\mathcal{T}_2$. Then $\mathcal{T}_1$ and $\mathcal{T}_3$ have a perspector X and mapping $(P, U) \mapsto X$ is called sqrtdiv. We have formula :

$$x : y : z = \frac{p^2}{u} : \frac{q^2}{v} : \frac{r^2}{w}$$

Definition 3.9.2. sqrtmul(U,X). The inverse mapping of *sqrtdiv* should be the mapping *sqrtmul* $(U, X) \mapsto P$ "defined" by :

$$p : q : r = \pm\sqrt{ux} : \pm\sqrt{vy} : \pm\sqrt{wz}$$

... but (1) the solution is not unique and (2) in fact, the problem cannot be stated clearly.

3.10 Danneels perspectors

Definition 3.10.1. First Danneels perspector. Let $\mathcal{T}_1 = A_U B_U C_U$ be the cevian triangle of a point $U = u : v : w$. Let L_A be the line through A parallel to $B_U C_U$, and define L_B and L_C cyclically. The lines L_A, L_B, L_C determine a triangle $\mathcal{T}_2$ perspective to $\mathcal{T}_1$ (and in fact homothetic, with factor 2). The corresponding perspector is $\mathrm{DP}_1(U)$, the first Danneels perspector (#11037) of U. Using barycentrics :

$$\mathrm{DP}_1(U) = u^2(v+w) : v^2(w+u) : w^2(u+v)$$

Proof. Compute (from left to right) the row $B_U \wedge C_U \wedge \mathcal{L}_\infty \wedge A$ and cyclically. Obtain a matrix describing a trigon and takes the adjoint to obtain $\mathcal{T}_2$. Then compute the perspector. One can also remark that $\mathcal{T}_2$ is the anticevian triangle of :

$$X = u(v+w) : v(u+w) : w(u+v)$$

and obtain $\mathrm{DP}_1(U)$ as *cevadiv* (U/X) (homothetic property is obvious... and useless to compute the perspector). □

Remark 3.10.2. This point is named $D(U)$ in ETC. No name of only one letter! Moreover this conflicts with the Maple's derivation operator.

Proposition 3.10.3. *Point $G = X(2)$ is invariant under DP_1. Moreover, G, U and* $\mathrm{DP}_1(U)$ *are ever collinear. For example, Euler line is globally invariant.*

Proof. Check that $\det(X_2, U, \mathrm{DP}_1(X_2 + \lambda U)) = 0$. □

Proposition 3.10.4. *When* $\mathrm{DP}_1(X) = G$ *then either* $X = G$ *or* X *lies on the Steiner circumellipse.*

Proof. Write $\mathrm{DP}_1(X) = k\,G$ and solve. Except from $X = G$, $xy + yz + zx = 0$ is obtained. ☐

Proposition 3.10.5. *When* $\mathrm{DP}_1(U) \neq G$*, i.e. when* $U \neq G$ *and* U *not on the circumSteiner, it exists two other points that verify* $\mathrm{DP}_1(X) = \mathrm{DP}_1(U)$*. Using barycentrics, we have :*

$$\begin{aligned} X &\simeq \left(\begin{array}{c} v + w - 2\,u - W\,\frac{1}{(u+w)(u+v)} \\ w + u - 2\,v - W\,\frac{1}{(u+v)(v+w)} \\ u + v - 2\,w - W\frac{1}{(u+w)(v+w)} \end{array} \right) \qquad \text{where} \\ W^2 &= (u+v)(v+w)(w+u)\left(u^2(v+w) + v^2(w+u) + w^2(u+v) - 6\,uvw\right) \end{aligned}$$

Proof. Direct computation, assuming $xy + yz + zx \neq 0$. The main difficulty is to re-obtain a symmetrical expression after elimination of k, z and resolution on y. ☐

Example 3.10.6. Here is a list of pairs (I, J) of named points such that $\mathrm{DP}_1(X(I)) = X(J)$:

1	42	20	3079	189	1422	664	2	903	2	2481	2
2	2	25	3080	190	2	666	2	1113	25	2966	2
3	418	30	3081	264	324	668	2	1114	25	3225	2
4	25	69	394	290	2	670	2	1121	2	3226	2
5	3078	75	321	366	367	671	2	1370	455	3227	2
6	3051	99	2	648	2	886	2	1494	2	3228	2
7	57	100	55	651	222	889	2	2479	2		
8	200	110	184	653	196	892	2	2480	2		

Definition 3.10.7. Second Danneels perspector. Suppose $\mathcal{T}_1 = A_U B_U C_U$ is the cevian triangle of a point U. Let L_{AB} be the line through B parallel to $A_U B_U$, and let L_{AC} be the line through C parallel to $A_U C_U$. Define $A' = L_{AB} \cap L_{AC}$ and B', C' cyclically. Finally obtain $A" = BB' \cap CC'$ and $B"$, $C"$ cyclically. It happens that triangle $\mathcal{T}_2 = A"B"C"$ is perspective to $\mathcal{T}_1 = A_U B_U C_U$. The corresponding perspector is $\mathrm{DP}_2(U)$, the second Danneels perspector of U (Danneels, 2006). Using barycentrics :

$$\mathrm{DP}_2(U) = u(v-w)^2 : v(w-u)^2 : w(u-v)^2$$

Proof. Compute (left to right) $L_{AB} = A_U \wedge B_U \wedge \mathcal{L}_\infty \wedge B$, L_{AC} accordingly, then $A' = L_{AB} \wedge L_{AC}$ and B', C' cyclically. Obtain $A" = (B \wedge B') \wedge (C \wedge C')$ and $B"$, $C"$cyclically. See that $\mathcal{T}_2 = A"B"C"$ is the anticevian triangle of point :

$$X \doteq u(v-w) : v(w-u) : w(u-v)$$

and obtain $\mathrm{DP}_2(U)$ as *cevadiv* (U, X). ☐

Proposition 3.10.8. *The circumconic through* $A, B, C, U, isot(U)$ *admits* $\mathrm{DP}_2(U)$ *as center and*

$$u\left(v^2 - w^2\right) : v\left(w^2 - u^2\right) : w\left(u^2 - v^2\right)$$

as perspector. And therefore isotomic conjugates have the same second Danneels' perspector.

Proof. Immediate computation. ☐

Example 3.10.9. List of $(U, U^*, \mathrm{DP}_2(U))$:

U	U^*	DP	U	U^*	DP	U	U^*	DP
1	75	244	37	274	3121	394	2052	3269
3	264	2972	42	310	3122	519	903	1647
4	69	125	57	312	2170	524	671	1648
6	76	3124	81	321	3125	536	3227	1646
7	8	11	94	323	2088	538	3228	1645
9	85	3119	98	325	868	2394	2407	1637
10	86	3120	99	523	1649	2395	2396	2491
20	253	122	100	693	3126	2397	2401	3310
30	1494	1650	200	1088	2310	2398	2400	676

and of points without named isotomic :

$$[43, 3123], [88, 2087], [694, 2086], [1022, 1635], [1026, 2254], [2421, 3569]$$

Chapter 4

The French touch

4.1 The random observer

When dealing with triangle geometry, we have to organize the coexistence of three kinds of objects. We have vectors, we have points and we have 3-tuples. A vector describes a translation of the "true plane". A vector has a direction but also a length and therefore is not defined "up to a proportionality factor". The set of all these vectors is a 2-dimensional vector space $\mathcal{V}$ (more about it in what follows). A point is either an element of the "true plane", i.e. an ordinary point at finite distance, or a point at infinity describing the direction of some line. Such points have therefore to be described "up to a proportionality factor" by a column that belongs to a given copy of $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$. In the same vein, lines are described "up to a proportionality factor" by a row that belongs to another copy $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$.

And we need to talk with our computer in order to let it compute all the required results. These computations are done using 3-tuples and tools acting over 3-tuples so that computations must be described using $\mathbb{R}^3$ rather than $\mathcal{V}$ or $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$. How to organize the coexistence of these three points of view is the object of a well known theorem.

Theorem 4.1.1. *Any affine space $\mathcal{E}$ can be embedded into a vector space $\widehat{E}$ in such a way that $\mathcal{E}$ becomes an affine hyperplane of $\widehat{E}$.*

Proof. By definition, $\mathcal{E}$ is not empty. Choose $\sun \in \mathcal{E}$ (the random observer) and write $\mathcal{E} = \sun + \mathcal{V}$ where $\mathcal{V}$ is the vector space associated with $\mathcal{E}$. Choose $\smiley \notin \mathcal{E}$ and write $E = \mathcal{V} \oplus \mathbb{R}\overrightarrow{\smiley\sun}$. Define ζ as the last coordinate in vector space E. Then $\mathcal{E} = \{m \in E \mid \zeta(m) = 1\}$. Moreover the vector hyperplane $\{m \in E \mid \zeta(m) = 0\}$ is an isomorphic copy of $\mathcal{V}$. $\square$

Notation 4.1.2. An affine description of a point at finite distance P is a 3-tuple (ξ, η, ζ) where $\zeta = 1$ is assumed. The semantic of these coordinates is the pre-existence of some random observer, that uses a Cartesian frame (ξ, η) to describe what is occurring before her eyes.

Let us take an example and begin with an informal approach (cf `http://www.les-mathematiques.net/phorum/read.php?8,585414,586289#msg-586289`). We have a point P, given by a column, and two triangles T_1, T_2 given by the columns of their vertices :

$$P = \frac{1}{1183}\begin{pmatrix} 252 + 130\sqrt{35} \\ 105 - 312\sqrt{35} \\ 1183 \end{pmatrix}, \quad T_1 = \frac{1}{169}\begin{pmatrix} 1284 & -744 & 40\sqrt{35} \\ 535 & -310 & -96\sqrt{35} \\ 169 & 169 & 169 \end{pmatrix}$$

$$T_2 = \frac{1}{1690}\begin{pmatrix} -9000 + 325\sqrt{35} & 14400 + 520\sqrt{35} & 5040 - 2600\sqrt{35} \\ -3750 - 780\sqrt{35} & 6000 - 1248\sqrt{35} & 2100 + 6240\sqrt{35} \\ 1690 & 1690 & 1690 \end{pmatrix}$$

As it should be, each column verifies $\zeta = 1$, which is the equation of the affine plane $\mathcal{E}$ when seen as a subspace of $\mathbb{R}^3$.

We define $\boxed{W}$ as the matrix that transforms the matrix T of a triangle (P_j) into the matrix of the sideline vectors $\left(\overrightarrow{P_{j+1}P_{j+2}}\right)$ of this triangle (indices are taken modulo 3 so that $P_4 = P_1$ etc). And now, we compute $\boxed{\mathcal{K}} = {}^{t}\boxed{W} \cdot {}^{t}T \cdot T \cdot \boxed{W}$ for both triangles. We have :

$$\boxed{W} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\boxed{\mathcal{K}_1} = \begin{pmatrix} 36 & 26 & -62 \\ 26 & 81 & -107 \\ -62 & -107 & 169 \end{pmatrix} = \begin{pmatrix} a^2 & -S_c & -S_b \\ -S_c & b^2 & -S_a \\ -S_b & -S_a & c^2 \end{pmatrix}$$

$$\boxed{\mathcal{K}_2} = \begin{pmatrix} 4212/5 & -702 & -702/5 \\ -702 & 3159/4 & -351/4 \\ -702/5 & -351/4 & 4563/20 \end{pmatrix} = \begin{pmatrix} \alpha^2 & -S_\gamma & -S_\beta \\ -S_\gamma & \beta^2 & -S_\alpha \\ -S_\beta & -S_\alpha & \gamma^2 \end{pmatrix}$$

Here we use a, b, c for the sidelengths of the first triangle and α, β, γ for the second one. The Conway's symbols $S_a = \left(b^2 + c^2 - a^2\right)/2$ etc. are defined accordingly. The letter used to name the matrix $\boxed{\mathcal{K}}$ has been chosen by reference to the Al-Kashi formula. Clearly, row $(1, 1, 1)$ belongs to the kernel of matrix $\boxed{\mathcal{K}}$. The characteristic polynomial of this matrix is $\mu^3 - \left(a^2 + b^2 + c^2\right)\mu^2 + 12\,S^2\mu$. One eigenvalue is $\mu = 0$ and the other two are real (from symmetry of $\boxed{\mathcal{K}}$) and positive.

Remark 4.1.3. Matrix $\boxed{\mathcal{M}}$ used to compute the orthodir of a line is nothing but $\boxed{\mathcal{M}} = \boxed{\mathcal{K}}/2S$.

A more "stratospheric", but nevertheless equivalent, definition for these Al-Kashi matrices would be :

$$\boxed{\mathcal{K}} \doteq {}^t\boxed{W}\,.\,{}^tT.\boxed{Pyth_3}.T.\boxed{W}$$

where $\boxed{Pyth_3}$ describes any $\mathbb{R}^3$-quadratic form that embeds $(\xi, \eta) \mapsto \xi^2 + \eta^2$, the quadratic form of the ordinary affine Euclidean plane. This embedding quadratic form depends on three arbitrary parameters since 6 coefficients are needed for dimension three, while only 3 are needed for dimension two.

4.2 An involved observer

Now, we will describe how things are looking when the observer is no more a random ☼ but rather an actor of the play. For example, we can take triangle T_1 as a new vector basis inside vector space $\mathbb{R}^3$ and calculate everything again using this new basis. From :

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

it is clear that condition $\zeta = 0$ that shows that a 3-tuple belongs to $\mathcal{V}$ becomes now $x + y + z = 0$. Defining $\mathcal{L}_\infty = (1, 1, 1)$ this can be rewritten as $\mathcal{L}_\infty\,.\,{}^t(x, y, z) = 0$. Seen "up to a proportionality factor" this will gives $\mathcal{L}_\infty\,.\,(x : y : z) = 0$, i.e. the condition for $x : y : z \in \mathbb{P}_\mathbb{R}\left(\mathbb{R}^3\right)$ to belong to the line at infinity. But this is not our purpose for the moment.

The $\mathbb{R}^3$-metric is now described by matrix ${}^tT_1\,.\,\boxed{Pyth_3}.T_1$. This matrix depends in turn on three arbitrary parameters. In fact, any other matrix that can be written as :

$${}^tT_1\,.\,\boxed{Pyth_3}.T_1 + U.\mathcal{L}_\infty + {}^t(U.\mathcal{L}_\infty)$$

using an arbitrary column U will be just as well to calculate the Pythagoras of $\mathcal{V}$-vector. A zero diagonal gives a more nice looking matrix, and is also more efficient for computing. Therefore we define the matrix $\boxed{Pyth}$ by this property and we obtain :

$${}^tT_1\,.\,\boxed{Pyth_3}.T_1 = \frac{1}{169}\begin{pmatrix} 11618 & -6465 & 169 \\ -6465 & 4013 & 169 \\ 169 & 169 & 2409 \end{pmatrix}, \quad \boxed{Pyth} = -\frac{1}{2}\begin{pmatrix} 0 & 169 & 81 \\ 169 & 0 & 36 \\ 81 & 36 & 0 \end{pmatrix}$$

Now, matrix $\boxed{\mathcal{K}_1}$ can be computed using :

$$\boxed{\mathcal{K}_1} = {}^t\boxed{W} . \boxed{Pyth} . \boxed{W} = \begin{pmatrix} 36 & 26 & -62 \\ 26 & 81 & -107 \\ -62 & -107 & 169 \end{pmatrix}$$

while the coordinates of the other triangle and the extra point are transformed according to :

$$\mathcal{T}_2 = T_1^{-1} . T_2 = \frac{1}{80} \begin{pmatrix} -30 & 48 & 240 \\ 45 & -72 & 360 \\ 65 & 104 & -520 \end{pmatrix}, \quad P_{[1]} = T_1^{-1} . P = \frac{1}{28} \begin{pmatrix} 6 \\ 9 \\ 13 \end{pmatrix}$$

And now, matrix $\boxed{\mathcal{K}_2}$ is computed using :

$$\boxed{\mathcal{K}_2} = {}^t\boxed{W} . {}^t\mathcal{T}_2 . \boxed{Pyth} . \mathcal{T}_2 . \boxed{W} = \begin{pmatrix} 4212/5 & -702 & -702/5 \\ -702 & 3159/4 & -351/4 \\ -702/5 & -351/4 & 4563/20 \end{pmatrix}$$

The quadratic form $\boxed{Pyth_2}$ can be obtained directly from $\boxed{Pyth}$, using formula :

$$\boxed{Pyth_2} = {}^t\mathcal{T}_2 . \boxed{Pyth} . \mathcal{T}_2 + +U.\mathcal{L}_\infty + {}^t(U.\mathcal{L}_\infty)$$

and thereafter choosing U in order to get a zero diagonal. We obtain :

$$\boxed{Pyth_2} = \begin{pmatrix} 0 & -4563/40 & -3159/8 \\ -4563/40 & 0 & -2106/5 \\ -3159/8 & -2106/5 & 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & \gamma^2 & \beta^2 \\ \gamma^2 & 0 & \alpha^2 \\ \beta^2 & \alpha^2 & 0 \end{pmatrix}$$

4.3 From an involved observer to another one

All the preceeding computations are standard ones. As long as points have no dependence relations all together, nothing else can be done. On the contrary, when points are constructed from each other, another point of view is more powerful. Taking ABC as reference triangle, and describing our points in $\mathbb{P}_\mathbb{R}\left(\mathbb{R}^3\right)$, we obtain $P = 6 : 9 : 13$, $A' = -6 : 9 : 13$, $B' = 6 : -9 : 13$, $C' = 6 : 9 : -13$. If this only a random coincidence, nothing more can be said.

On the contrary, if these points are really defined as $P = a : b : c$, $A' = -a : b : c$, $B' = a : -b : c$, $C' = a : b : -c$, then things get more interesting. We have :

$$P_{[2]} = b + c - a : c + a - b : a + b - c$$

$$\boxed{\mathcal{K}_2} = \begin{pmatrix} \dfrac{4\,a^2bc}{(a-b+c)\,(b+a-c)} & \dfrac{-2\,abc}{b+a-c} & \dfrac{-2\,abc}{a-b+c} \\ \dfrac{-2\,abc}{b+a-c} & \dfrac{4\,ab^2c}{(b+c-a)\,(b+a-c)} & \dfrac{-2\,abc}{b+c-a} \\ \dfrac{-2\,abc}{a-b+c} & \dfrac{-2\,abc}{b+c-a} & \dfrac{4\,c^2ab}{(b+c-a)\,(a-b+c)} \end{pmatrix}$$

so that :

$$\left\{\alpha^2 = k\,\frac{4\,a^2bc}{(a-b+c)\,(b+a-c)}, \text{ etc}\right\}$$

Since barycentrics are defined only "up to a proportionality factor", the former set of equations has not to be solved explicitely. Only the ratios between the quantities a, b, c are needed. And this is simple to obtain. One gets :

$$(a : b : c) = \left(\alpha^2\left(\beta^2 + \gamma^2 - \alpha^2\right) : \beta^2\left(\gamma^2 + \alpha^2 - \beta^2\right) : \gamma^2\left(\beta^2 + \alpha^2 - \gamma^2\right)\right)$$

so that :

$$P_{[2]} = \frac{1}{\beta^2 + \gamma^2 - \alpha^2} : \frac{1}{\gamma^2 + \alpha^2 - \beta^2} : \frac{1}{\alpha^2 + \beta^2 - \gamma^2}$$

This results identifies the incenter of the reference triangle ABC as the orthocenter of the excentral triangle. In the same vein, the circumcenter of ABC can be identified as the nine-points center of $A'B'C'$. More details on this specific relation will be given in Subsection 17.2.4.

Chapter 5

Euclidean structure using barycentrics

Barycentric coordinates were intended to describe affine properties, i.e. properties that remains when points are moved freely. Therefore describing Euclidean properties when using barycentrics is often presented as contradictory. We will show that, on the contrary, all the required properties can be described simply. The key fact is that orthogonality only depends on the directions of lines so that all what is really needed is a bijection that sends each point at infinity onto the point that characterizes the orthogonal direction.

5.1 Lengths and areas

Notation 5.1.1. In this section, T_0 describes the reference triangle ABC in the affine plane, and T describes a generic triangle P_j (indices are dealt modulo 3, i.e. $P_4 = P_1$, etc. In other words :

$$T_0 = \begin{vmatrix} \xi_a & \xi_b & \xi_c \\ \eta_a & \eta_b & \eta_c \\ 1 & 1 & 1 \end{vmatrix}, \; T = \begin{vmatrix} \xi_\alpha & \xi_\beta & \xi_\gamma \\ \eta_\alpha & \eta_\beta & \eta_\gamma \\ 1 & 1 & 1 \end{vmatrix} \tag{5.1}$$

We will use $|BC| = a$, $|P_2P_3| = \alpha$, etc together with $S_a = \left(b^2 + c^2 - a^2\right)/2$, $S_\alpha = \left(\beta^2 + \gamma^2 - \alpha^2\right)/2$, etc. In other words,

$$(\xi_a - \xi_b)^2 + (\eta_a - \eta_b)^2 = c^2, \text{ etc} \tag{5.2}$$

Lemma 5.1.2. *Let $\mathcal{T}$ be the matrix describing triangle (P_j) wrt triangle ABC, i.e. the matrix whose columns are the barycentrics $p_i : q_i : r_i$ of the P_i. Then :*

$$\mathcal{T} = T_0^{-1} . T . \begin{pmatrix} p_1 + q_1 + r_1 & 0 & 0 \\ 0 & p_2 + q_2 + r_2 & 0 \\ 0 & 0 & p_3 + q_3 + r_3 \end{pmatrix}$$

Definition 5.1.3. Matrix W. The matrix $\boxed{W}$ is defined by :

$$\boxed{W} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{5.3}$$

Proposition 5.1.4. *When a matrix T, as defined in (5.1), gives the vertices P_j of a triangle, then $T . \boxed{W}$ gives the sideline vectors $\overrightarrow{P_{j+1}P_{j+2}}$ of this triangle (using modulo 3 indices). On the other hand, matrix $\boxed{W}$ can be used to compute the point at infinity $U \in \mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$ of a line given by its barycentrics Δ. In this case, we have :*

$$U = \Delta \wedge \mathcal{L}_\infty = \boxed{W} . \left({}^t\Delta\right) \simeq {}^t\left(\Delta . \boxed{W}\right)$$

Proof. The first part is obvious from $\overrightarrow{P_jP_{j+1}} = P_{j+1} - P_j$ that holds when P_j are 3-tuples in the $\zeta = 1$ plane. The second part is the very definition of the $\wedge$ operator. □

Lemma 5.1.5. ***Matrix K.*** *We have the following Al-Kashi formula :*

$$\boxed{\mathcal{K}} = {}^t\boxed{W} \,.\, {}^tT \,.\, T \,.\, \boxed{W} = \begin{pmatrix} \alpha^2 & -S_\gamma & -S_\beta \\ -S_\gamma & \beta^2 & -S_\alpha \\ -S_\beta & -S_\alpha & \gamma^2 \end{pmatrix}$$

Proof. Diagonal elements are $\left\langle \overrightarrow{BC} \mid \overrightarrow{BC} \right\rangle$, etc and the others are $\left\langle \overrightarrow{BC} \mid \overrightarrow{CA} \right\rangle = -\left\langle \overrightarrow{CB} \mid \overrightarrow{CA} \right\rangle$, etc. □

Proposition 5.1.6. *Let matrix* $\boxed{\mathcal{T}}$ *define a (finite) triangle* $\mathcal{T}$ *with vertices* $P_i = p_i : q_i : r_i$*. Then area of* $\mathcal{T}$ *is given by :*

$$S \times \frac{\det \boxed{\mathcal{T}}}{\prod (p_i + q_i + r_i)} \tag{5.4}$$

while the area S *of the reference triangle is given by the Heron formula :*

$$S^2 = \frac{1}{16}(a+b+c)(b+c-a)(c+a-b)(a+b-c) \tag{5.5}$$

Proof. The first formula is obvious from Lemma 5.1.2 and the well-known formula

$$S = \frac{1}{2} \begin{vmatrix} \xi_a & \xi_b & \xi_c \\ \eta_a & \eta_b & \eta_c \\ 1 & 1 & 1 \end{vmatrix} \tag{5.6}$$

that gives the oriented area of a triangle. The denominator of (5.4) enforces the required invariance wrt multiplicative factors acting on barycentrics, and recognizes the fact that only triangles with finite vertices have an area.

As it should be, this formula acknowledges that area of ABC is S. The Heron formula can be proved in many ways, one of them being :

$$4S^2 = |AB|^2 \, |AC|^2 \sin^2 A = b^2c^2 - \left\langle \overrightarrow{AB} \mid \overrightarrow{AC} \right\rangle^2 = b^2c^2 - S_a^2$$

□

Remark 5.1.7. A key point is that formula (5.4) is of first degree in S : once the orientation of the reference triangle is chosen, all other orientations are fixed.

5.2 Embedded Euclidean vector space

Definition 5.2.1. When points P, Q at finite distance are given by their barycentrics $p : q : r$ and $u : v : w$, the **embedded vector** from P to U is defined as :

$$\overrightarrow{vec}(P,\, U) = \frac{1}{u+v+w} \begin{pmatrix} u \\ v \\ w \end{pmatrix} - \frac{1}{p+q+r} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix} \tag{5.7}$$

Proposition 5.2.2. *Embedded vectors belong to the vector plane* $\mathcal{V} : x + y + z = 0$*, seen as a subspace of the Cartesian (non projective) vector space* $\mathbb{R}^3$*. These vectors obey to the usual Chasles rule :*

$$\overrightarrow{vec}(P_1,\, P_3) = \overrightarrow{vec}(P_1,\, P_2) + \overrightarrow{vec}(P_2,\, P_3)$$

and space $\mathcal{V}$ *is isomorph to the usual vector space* $\mathbb{R}^2$ *where* $\overrightarrow{P_1P_2} = (\xi_2 - \xi_1,\, \eta_2 - \eta_{1a})$.

Proof. Obvious from definition. Nevertheless, a touch stone for the sequel. Defined this way, vector space $\mathcal{V}$ is how the plane $\zeta = 0$ is perceived when using triangle ABC as reference triangle. □

Proposition 5.2.3. *When using (ξ, η, ζ) coordinates, the metric of the usual Euclidean plane $\mathcal{V}$ is described by matrix :*

$$\boxed{Pyth_3} \doteq \left(\begin{array}{ccc} 1 & 0 & * \\ 0 & 1 & * \\ * & * & * \end{array}\right) \tag{5.8}$$

where $*$ are placeholders for three arbitrary parameters. When using barycentrics (wrt triangle ABC), this matrix is replaced by ${}^tT_0 \,.\, \boxed{Pyth_3} . T_0$, that depends also on three arbitrary parameters. In fact, any other matrix that can be written as :

$${}^tT_0 \,.\, \boxed{Pyth_3} \,.\, T_0 + U \,.\, \mathcal{L}_\infty + {}^t(U \,.\, \mathcal{L}_\infty) \tag{5.9}$$

can be used to define the metric of vector space $\mathcal{V}$.

Proof. This is the usual formula for a change of basis in a bilinear form. Independence from column U comes from $\mathcal{L}_\infty \,.\, \mathcal{V} = 0$. □

Theorem 5.2.4. Pythagoras theorem. *Define the Pythagoras bilinear form by : $\mathbb{R}^3 \times \mathbb{R}^3 \hookrightarrow \mathbb{R}$: $\langle \overrightarrow{vec_1} \mid \overrightarrow{vec_2} \rangle = {}^t\overrightarrow{vec_1} \,.\, \boxed{Pyth} \,.\, \overrightarrow{vec_2}$ where :*

$$\boxed{Pyth} = \frac{1}{2}\left(\begin{array}{ccc} 0 & -c^2 & -b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{array}\right) \tag{5.10}$$

When restricted to $\mathcal{V}$, this form extends the transformation $\mathbb{R}^2 \hookrightarrow \mathcal{V}$: $\overrightarrow{PQ} \mapsto \overrightarrow{vec}(P, Q)$ into an isomorphism of Euclidean spaces. Therefore, the squared distance between two (finite) points of the triangle plane is given by :

$$\begin{aligned} |PU|^2 &= {}^t\overrightarrow{vec}(P, U) \,.\, \boxed{Pyth} \,.\, \overrightarrow{vec}(P, U) \\ \left|\overrightarrow{(\rho, \sigma, \tau)}\right|^2 &= -\left(a^2\sigma\tau + b^2\tau\rho + c^2\rho\sigma\right) \end{aligned} \tag{5.11}$$

Proof. Matrix $\boxed{Pyth}$ is what is obtain when choosing column U in (5.9) in order to obtain a zero diagonal instead of using $\boxed{\mathcal{K}}$ directly. A less "stratospheric" proof is direct inspection. We have :

$$\langle \overrightarrow{vec}(A, B) \mid \overrightarrow{vec}(A, B) \rangle = c^2 \quad (\text{etc}) \quad \text{and} \quad \langle \overrightarrow{vec}(A, B) \mid \overrightarrow{vec}(A, C) \rangle = S_a \quad (\text{etc})$$

Since $S_a = bc \cos A$, property holds for a basis. And linearity extends the result to all the other vectors. □

Remark 5.2.5. Using the circumcircle formula (5.13) obtained in the next subsection, we have

$$\left|\overrightarrow{(\rho, \sigma, \tau)}\right|^2 = -\Gamma(\rho, \sigma, \tau)$$

A greater attention to this relation will be given in Section 11.2.

5.3 About circumcircle and infinity line

Definition 5.3.1. The **power** of a point $X = x : y : z$ (at finite distance) with respect to the circle Ω centered at P with radius R is :

$$power(\Omega, X) \doteq |PX|^2 - R^2$$

Theorem 5.3.2. *The* ***power formula*** *giving the Ω-power of any point $X = x : y : z$ from the power at the three vertices of the reference triangle is :*

$$\begin{aligned} power(\Omega, X) &= \frac{ux + vy + wz}{x + y + z} - \frac{a^2yz + b^2xz + c^2xy}{(x + y + z)^2} \\ \text{where } u &= power(\Omega, A), \text{ etc} \end{aligned} \tag{5.12}$$

Proof. Use (5.7) to obtain $\overrightarrow{PX}$ and then Theorem 5.2.4 to obtain $power\left(\Omega,\, X\right)$. Substitute $y = z = 0$ to obtain u, etc. Then a simple substraction leads to the required result. □

Proposition 5.3.3. *The equation of any circle can be written as :*

$$\begin{aligned}\Omega\left(x,y,z\right) \doteq \Gamma\left(x,y,r\right) - \left(x+y+z\right)\left(ux+vy+wz\right) &= 0\\ \text{where } u = power\left(\Omega,\, A\right),\text{ etc}\end{aligned}$$

where $x:y:z$ is the arbitrary point and Γ is the standard equation of the circumcircle, defined as :

$$\Gamma\left(x,y,z\right) \doteq a^2yz + b^2xz + c^2xy = 0 \tag{5.13}$$

Proof. Obvious from (5.12) and $power\left(\Gamma,\, A\right) = 0$, etc. □

Remark 5.3.4. Both equations $power\left(\Omega, X\right) = 0$ and $\Omega\left(X\right) = 0$ don't define the same object ! The first one doesn't contains points at infinity, while the other one contains the umbilics. For more details, see Chapter 10.

Corollary 5.3.5 (Heron)**.** *Center and radius of the circumcircle are :*

$$\begin{aligned}X_3 &= a^2\left(b^2+c^2-a^2\right) : b^2\left(c^2+a^2-b^2\right) : c^2\left(a^2+b^2-c^2\right)\\ R^2 &= \frac{a^2b^2c^2}{\left(a+b+c\right)\left(a+b-c\right)\left(b+c-a\right)\left(c+a-b\right)}\end{aligned}$$

Computed Proof. Direct elimination from $\left\{\left|XA\right|^2 = R^2,\text{ etc}\right\}$ and (5.7,5.11). □

Proposition 5.3.6. *For a line $\Delta = \rho : \sigma : \tau$, not the infinity line, point $Q = \Delta \wedge \mathcal{L}_\infty$ is on $\mathcal{L}_\infty$ while $U = isogon\left(Q\right)$ is on the circumcircle. Therefore, rational parameterizations of the line at infinity and the circumcircle are :*

$$\mathcal{L}_\infty = \left\{\sigma-\tau : \tau-\rho : \rho-\sigma \mid \rho:\sigma:\tau \neq \mathcal{L}_\infty\right\} \tag{5.14}$$

$$circumcircle = \left\{\frac{a^2}{\sigma-\tau} : \frac{b^2}{\tau-\rho} : \frac{c^2}{\rho-\sigma} \mid \rho:\sigma:\tau \neq \mathcal{L}_\infty\right\} \tag{5.15}$$

Proof. Point Q is the point at infinity of the line $\rho x + \sigma y + \tau z = 0$, $U \in \Gamma$ is obvious from (5.13) and bijectivity of $Q \mapsto U$ is clear. □

Remark 5.3.7. Information conveyed by a 3-tuple like (5.7) is multiple. A first part is the direction of line PU, described –up to a proportionality factor– by the point $\rho : \sigma : \tau \in \mathcal{L}_\infty$. Another part is the squared length $\left|PU\right|^2$ given by (5.11). In this formula, circumcircle appears as the conic that defines how lengths are computed in each direction.

5.4 Orthogonality

Proposition 5.4.1. *Given a 3-tuple $P = (p,q,r) \in \mathbb{R}^3$, but not on $\mathcal{V}$, it exists a linear transform ψ such that (i) $\psi\left(P\right) = 0$, (ii) $\psi\left(\mathcal{V}\right) = \mathcal{V}$ (iii) for all $V \in \mathcal{V}$, $\left\langle \psi\left(V\right) \mid \psi\left(V\right)\right\rangle = \left\langle V \mid V\right\rangle$ while $\left\langle \psi\left(V\right) \mid V\right\rangle = 0$. Then its characteristic polynomial is $\mu^3 + \mu$ and we have :*

$$Orth\left(P\right) = \frac{1}{2\left(p+q+r\right)S}\begin{pmatrix} qS_b - rS_c & -\left(ra^2+pS_b\right) & qa^2+pS_c\\ rb^2+qS_a & rS_c - pS_a & -\left(pb^2+qS_c\right)\\ -\left(qc^2+rS_a\right) & pc^2+rS_b & pS_a - qS_b\end{pmatrix}$$

The opposite of this matrix is the only other solution to the problem.

Computed Proof. Assertions $M\,.\,P = 0$, $\mathcal{L}_\infty\,.\,M = 0$ and $\left\langle \psi\left(V\right) \mid V\right\rangle = 0$ when $V = x : y : -x-y$ gives nine equations. Elimination leads to the given matrix. Then $\left\langle \psi\left(V\right) \mid \psi\left(V\right)\right\rangle = \left\langle V \mid V\right\rangle$ leads to the coefficient. Division by $p+q+$ renforces the fact that P is at finite distance. □

Corollary 5.4.2. *Matrices Orth (P) and Orth (U) relative to finite points P, U are related by the "translation" formula :*

$$Orth(P) = Orth\,(U)\ .\ \left(1 - \frac{1}{p+q+r}\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \mathcal{L}_\infty\right)$$

Proof. Acting at infinity, both matrices induces a quater-turn. At finite distance, it remains only to move the kernel to the right place. □

Theorem 5.4.3. ***Orthopoint****. A point at infinity $V \in \mathcal{L}_\infty$ defines a direction of lines. The point $W \in \mathcal{L}_\infty$ that defines the orthogonal direction is called the orthopoint of V. We have $W \simeq Orth\,(P)\ .U$ whatever the choice of P. Nevertheless, the choice O =X(3) is interresting since the Orth (O) matrix verifies :*

$$\boxed{OrtO} \doteq {}^t\boxed{W} . \boxed{Pyth} \div 2S$$

This ${}^t\boxed{W} . \boxed{Pyth}$ *matrix is independent of the U-choice appearing in (5.9), and we have :*

$$\boxed{OrtO} = \frac{1}{4S}\begin{pmatrix} c^2 - b^2 & -a^2 & a^2 \\ b^2 & a^2 - c^2 & -b^2 \\ -c^2 & c^2 & b^2 - a^2 \end{pmatrix} \tag{5.16}$$

Proof. Straightforward computation. We have the formal rule *orthopoint* $= \mathcal{L}_\infty \wedge {}^t(U \div_b X_4) \div (-4S)$ when $U \in \mathcal{L}_\infty$. □

Proposition 5.4.4. *Matrix* $\boxed{OrtO}$ *describes the* $+90°$ *rotation in the* $\mathcal{V}$ *space, while matrix* $\boxed{W} . \boxed{Pyth} \div 2S$ *describes the* $-90°$ *rotation.*

Proof. The only thing to prove is the orientation. Go back to the ordinary cartesian coordinates by(5.1), substitute the squared sidelengths with (5.2), and obtain :

$$\boxed{T_0} . \boxed{OrtO} . \boxed{T_0}^{-1} = \begin{pmatrix} 0 & -1 & * \\ 1 & 0 & * \\ 0 & 0 & * \end{pmatrix}$$

□

Remark 5.4.5. Any *Orth* (*P*) matrix describes a quater turn in the Euclidean plane $\mathcal{V}$. In order to provide a better perception of this result, let O =X(3), H =X(4), put $W_3^2 = 16\,|OH|^2\, S^2 = \sum_3 a^6 - \sum_6 a^4 b^2 + 3a^2b^2c^2$ and consider the linear transform whose matrix is $\phi = [X\,(3)\,,\ X\,(30)\,,\ X\,(523)]$. When computing $|OH|^2$, vector $\overrightarrow{OH}$ is involved and therefore X(30) (the direction of the Euler line), while X(523) is known to be the direction orthogonal to those of the Euler line. We have :

$$\phi = \begin{pmatrix} a^2\left(b^2 + c^2 - a^2\right) & 2\,a^4 - \left(b^2 - c^2\right)^2 - a^2\left(b^2 + c^2\right) & b^2 - c^2 \\ b^2\left(c^2 + a^2 - b^2\right) & 2\,b^4 - \left(c^2 - a^2\right)^2 - b^2\left(c^2 + a^2\right) & c^2 - a^2 \\ c^2\left(a^2 + b^2 - c^2\right) & 2\,c^4 - \left(a^2 - b^2\right)^2 - c^2\left(a^2 + b^2\right) & a^2 - b^2 \end{pmatrix}$$

$${}^t\phi . \boxed{Pyth} . \phi = \begin{pmatrix} -16\,a^2c^2b^2S^2 & 0 & 0 \\ 0 & 16\,S^2W_3^2 & 0 \\ 0 & 0 & W_3^2 \end{pmatrix}$$

$$\phi . \boxed{OrtO} . \phi^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (4S)^{-1} \\ 0 & 4S & 0 \end{pmatrix}$$

$${}^t\phi . {}^t\boxed{OrtO} . \boxed{Pyth} . \boxed{OrtO} . \phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 16\,S^2W_3^2 & 0 \\ 0 & 0 & W_3^2 \end{pmatrix}$$

Proposition 5.4.6. *The* **orthodir** U *of any line* Δ *(except from the line at infinity) is defined as the orthopoint of* $\Delta \wedge \mathcal{L}_\infty$*. It can be computed as* $U = \boxed{\mathcal{M}} \,.\, {}^t\Delta$ *where :*

$$\boxed{\mathcal{M}} \doteq \boxed{OrtO} \,.\, \boxed{W} = \frac{1}{2S}\, {}^t\boxed{W} \,.\, \boxed{Pyth} \,.\, \boxed{W} = \frac{1}{2S} \left(\begin{array}{ccc} a^2 & -S_c & -S_b \\ -S_c & b^2 & -S_a \\ -S_b & -S_a & c^2 \end{array} \right) \tag{5.17}$$

Proof. This comes directly from the orthopoint formula. Normalization factor $1/2S$ is useless here, but will be required in what follows. One can check that, for example, the first column gives the direction of the first altitude (orthogonal to sideline BC). □

Remark 5.4.7. Characteristic polynomial of $\boxed{\mathcal{M}}$ is : $\mu^3 + \mu^2 \left(a^2 + b^2 + c^2\right)/2/S + 3\mu$ and it can be checked that its left null space is $[1, 1, 1]$, the row associated with $\mathcal{L}_\infty$: for any column X, $\boxed{\mathcal{M}} \,.\, X \in \mathcal{L}_\infty$ (as it should be).

Remark 5.4.8. Adjoint of matrix $\boxed{\mathcal{M}}$ is the "all ones" matrix, i.e. ${}^t\mathcal{L}_\infty \,.\, \mathcal{L}_\infty$. As it should be, this matrix has rank one (since $\boxed{\mathcal{M}}$ has rank $n-1$), and product $Adjoint\left(\boxed{\mathcal{M}}\right) \,.\, \boxed{\mathcal{M}}$ is the null matrix.

Fact 5.4.9. *The triangle of the midpoints of the altitudes has barycentrics :*

$$\left(\begin{array}{ccc} a^2 & S_c & S_b \\ S_c & b^2 & S_a \\ S_b & S_a & c^2 \end{array} \right)$$

5.5 Angles between straight lines

Proposition 5.5.1. *Let* P, U_1, U_2 *be three points at finite distance, such that* $\overrightarrow{PU_i} \neq \overrightarrow{0}$*. Then :*

$$\begin{aligned} |PU_1| \,.\, |PU_2| \,.\, \sin\left(\overrightarrow{PU_1}, \overrightarrow{PU_2}\right) &= 2S\, \frac{\det(P, U_1, U_2)}{(p+q+r)\prod(u_i + v_i + w_i)} \\ |PU_1| \,.\, |PU_2| \,.\, \cos\left(\overrightarrow{PU_1}, \overrightarrow{PU_2}\right) &= 2S\, \frac{(P \wedge U_1) \,.\, \boxed{\mathcal{M}} \,.\, {}^t(P \wedge U_2)}{(p+q+r)^2 \prod(u_i + v_i + w_i)} \end{aligned}$$

Proof. The sin formula comes from (5.4), while the cos formula can be obtained by using (5.11) into $|PU_1|^2 + |PU_2|^2 - |U_1U_2|^2$ and rearranging. □

Theorem 5.5.2. *Let* P *be a point at finite distance, and* U_1, U_2 *two other points (at finite distance or not). Then the angle between straight lines* PU_1*,* PU_2 *is characterized by its* ***tangent****, according to :*

$$\tan\left(\overbrace{PU_1, PU_2}\right) = (p+q+r)\, \frac{\det(P, U_1, U_2)}{(P \wedge U_1) \,.\, \boxed{\mathcal{M}} \,.\, {}^t(P \wedge U_2)} \tag{5.18}$$

When U_1*,* U_2 *are at infinity, the angle between all the lines having the given directions can be computed as :*

$$\tan_\infty\left(\overbrace{U_1, U_2}\right) = \frac{2S\,(v_1 w_2 - w_1 v_2)}{(v_1 w_2 + w_1 v_2)\, S_a + w_1 w_2 b^2 + v_1 v_2 c^2}$$

Proof. The key point here is that formula (5.18) is square-root free. Extension to U_i at infinity is obtained by continuity after cancellation of the $(u_i + v_i + w_i)$. Formula $\tan_\infty$ is not formally symmetrical ($P = A$ has been used). But the u_i are nevertheless present since $u_i = -v_i - w_i$. □

Theorem 5.5.3. Tangent of two lines. *If the triangle plane is oriented according to* $\left(\overbrace{BC, BA}\right) = +A$*, then oriented angle from line* Δ_1 *to line* Δ_2 *is characterized by :*

$$\tan\left(\overbrace{\Delta_1, \Delta_2}\right) = \frac{\Delta_1 \,.\, \boxed{W} \,.\, {}^t\Delta_2}{\Delta_1 \,.\, \boxed{\mathcal{M}} \,.\, {}^t\Delta_2} \tag{5.19}$$

name	ψ	#		#	
$\mathcal{L}_\infty$	2		$[1;1;1]$	(12.8)	$[0;1;0]$
$\Delta \wedge \mathcal{L}_\infty$ $\boxed{W}$	3	(5.3)	$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	(12.9)	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix}$
pythagoras $\boxed{Pyth}$	5	(5.10)	$\dfrac{1}{2}\begin{pmatrix} 0 & -c^2 & -b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{pmatrix}$	(12.10)	$\dfrac{R^2}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
orthopoint $\boxed{OrtO}$	4	(5.16)	$\dfrac{1}{4S}\begin{pmatrix} c^2-b^2 & -a^2 & a^2 \\ b^2 & a^2-c^2 & -b^2 \\ -c^2 & c^2 & b^2-a^2 \end{pmatrix}$	(12.11)	$i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
orthodir $\boxed{\mathcal{M}}$	3	(5.17)	$\dfrac{1}{2S}\begin{pmatrix} a^2 & -S_c & -S_b \\ -S_c & b^2 & -S_a \\ -S_b & -S_a & c^2 \end{pmatrix}$	(12.12)	$-i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
tangent of $\left(\widehat{\Delta_1,\, \Delta_2}\right)$		(5.19)	$\dfrac{\Delta_1 \,.\, \boxed{W} \,.\, {}^t\Delta_2}{\Delta_1 \,.\, \boxed{\mathcal{M}} \,.\, {}^t\Delta_2}$	(12.13)	$\dfrac{\Delta_1 \,.\, \boxed{W_z} \,.\, {}^t\Delta_2}{\Delta_1 \,.\, \boxed{\mathcal{M}_z} \,.\, {}^t\Delta_2}$
$\boxed{OrtO} = \frac{1}{2S}\, {}^t\boxed{W} \,.\, \boxed{Pyth}$; ${}^t\boxed{OrtO} \,.\, \boxed{Pyth} \,.\, \boxed{OrtO} = \boxed{Pyth}$; $\boxed{\mathcal{M}} = \boxed{OrtO} \,.\, \boxed{W}$					
transform ψ : (1) point: $X \mapsto \boxed{Lu}\, X$; (2) line: $X \mapsto X \boxed{Lu^{-1}}$; (3) line to point $\boxed{Lu}\, X\, {}^t\boxed{Lu}$; (4) point to point $\boxed{Lu}\, X \boxed{Lu^{-1}}$; (5) quad form ${}^t\boxed{Lu^{-1}}\, X \boxed{Lu^{-1}}$					

Table 5.1: All these matrices

where $\boxed{W}$ *and* $\boxed{\mathcal{M}}$ *are exactly as given in* (5.3) *and* (5.17) *(i.e. not up to a proportionality factor).*

Proof. Simple use of $\Delta_1 \wedge \Delta_2$, $\mathcal{L}_\infty \wedge \Delta_1$, $\mathcal{L}_\infty \wedge \Delta_2$ in (5.18). Among other things, this formula tells us that $\vartheta = 0$ when each line contain the point at infinity of the other, while $|\vartheta| = \pi/2$ when each line contains the orthopoint of the other (formula is anti-symmetric). □

Stratospheric proof. . Start from the affine space $\mathcal{E}$. Equations of both lines are $a_j\xi + b_j\eta + c_j = 0$ and their angle is given by

$$\tan\left(\widehat{\Delta_1,\, \Delta_2}\right) = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} = \frac{\det\left(\mathcal{L}_3,\, D_1,\, D_2\right)}{D_1 \,.\, \boxed{Orth_3} \,.{}^t D_2}$$

where $\mathcal{L}_3 = [0, 0, 1]$ and $\boxed{Orth_3}$ is the matrix of quadratic form $\xi^2 + \eta^2$ –precisely this one, without any of the extra terms used in (5.8). Taking now ABC for basis, we have $\Delta_j = D_j \,.\, T_0^{-1}$, inducing a factor $1/2S$ in the numerator. Let us now compare the following two expressions :

$${}^t T_0^{-1} = \frac{1}{2S}\begin{bmatrix} \eta_b - \eta_c & -\eta_a + \eta_c & \eta_a - \eta_b \\ -\xi_b + \xi_c & \xi_a - \xi_c & -\xi_a + \xi_b \\ \xi_b\eta_c - \xi_c\eta_b & -\xi_a\eta_c + \xi_c\eta_a & \xi_a\eta_b - \xi_b\eta_a \end{bmatrix}$$

$$T_0 \,.\, \boxed{W} = \begin{bmatrix} -\xi_b + \xi_c & \xi_a - \xi_c & -\xi_a + \xi_b \\ -\eta_b + \eta_c & \eta_a - \eta_c & -\eta_a + \eta_b \\ 0 & 0 & 0 \end{bmatrix}$$

Due to the specific value of $\boxed{Orth_3}$, we can replace ${}^t T_0^{-1}$ by $T_0 \,.\, \boxed{W}/2S$ in the change of basis formulae and obtain $\boxed{Orth_3} \mapsto \boxed{\mathcal{K}}/\,(2S)^2$. Using the orthodir matrix instead of the Al-Kashi one leads to a formula without remaining factors. □

Remark 5.5.4. Rotations are examined at 14.2

5.6 Distance from a point to a line

Definition 5.6.1. The distance of point $P = p : q : r$ to line $\Delta = (\rho,\, \sigma,\, \tau)$ is the lower bound of the distance of P to a point U that belongs to Δ. By continuity, this bound is attained and is equal to the distance of P to its orthogonal projection P_0 on Δ.

Theorem 5.6.2. *Distance from point $P = p : q : r$ to line $\Delta = (\rho,\, \sigma,\, \tau)$ is given by :*

$$\operatorname{dist}(P,\, \Delta) = \sqrt{2S}\, \frac{\Delta \,.\, P}{(\mathcal{L}_\infty \,.\, P)\, \sqrt{\Delta \,.\, \boxed{\mathcal{M}} \,.{}^t\Delta}} \tag{5.20}$$

where $\mathcal{L}_\infty = [1, 1, 1]$ and $\boxed{\mathcal{M}}$ is as given in (5.17) (not up to a proportionality factor).

Proof. One obtains easily :

$$|P\, P_0|^2 = \frac{4S^2\, (\rho\, p + \sigma\, q + \tau\, r)^2}{(p+q+r)^2\, (a^2\rho^2 + b^2\sigma^2 + c^2\tau^2 - 2\,\sigma\,\tau\, S_a - 2\,\rho\,\tau\, S_b - 2\,\rho\,\sigma\, S_c)}$$

where each factor is recognizable (cf Theorem 5.5.3). □

Remark 5.6.3. Formula (5.20) is invariant when barycentrics of P or Δ are modified by a proportionality factor. Denominators are enforcing the fact that P is supposed to be finite, and Δ is not supposed to be the infinity line. The square root is the operator norm of the application $P \mapsto (\rho\, p + \sigma\, q + \tau\, r) \,/\, (x + y + z)$. And finally $\sqrt{2S}$ is the square root of the standard area, i.e. the standard unit of length.

5.7 Brocard points and the sequel

Proposition 5.7.1. Brocard points. *It exists exactly one point ω^+ and one point ω^- such that :*

$$\begin{array}{ccccc} \angle\,(A\omega^+,\, AC) & = & \angle\,(B\omega^+,\, BA) & = & \angle\,(C\omega^+,\, CB) \\ \angle\,(AB,\, A\omega^-) & = & \angle\,(BC,\, B\omega^-) & = & \angle\,(CA,\, C\omega^-) \end{array}$$

They are given by $\omega^+ = a^2b^2 : b^2c^2 : c^2a^2$ and $\omega^- = c^2a^2 : a^2b^2 : b^2c^2$. Moreover, when defined exactly *that way, both angles are equal. This quantity is called the Brocard angle and one has :*

$$\cot\omega = \cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4S} \tag{5.21}$$

Proof. Equating the tangents of the angles and eliminating, one obtains a third degree equation with one simple real root and tho others that involves $\sqrt{-S^2}$. As it should be Brocard points are isogonal conjugates of each other. The cot formula is given in Volenec (2005). □

Remark 5.7.2. The ETC points in the Brocard line are 9 (Mittenpunkt), 512 (at infinity), 881, 882, 2524, 2531.

Lemma 5.7.3. *Since* $\tan\omega > 0$ *and* $|\omega| \leq \pi/6$*, we have :*

$$\cos(\omega) = \frac{c^2 + a^2 + b^2}{2\sqrt{a^2b^2 + a^2c^2 + b^2c^2}}\,,\ \sin(\omega) = \frac{2S}{\sqrt{a^2b^2 + a^2c^2 + b^2c^2}}$$

Fact 5.7.4. *Each Brocard point is at the intersection of three isogonal circles, according to :*

$$\begin{array}{ccc} \angle\left(\omega^+B,\, \omega^+C\right) & = & \angle\,(BA,\, BC) \\ \angle\left(\omega^-B,\, \omega^-C\right) & = & \angle\,(CB,\, CA) \end{array}$$

Proposition 5.7.5. Neuberg circles. *The locus of A when B, C, ω are given is a circle (Neuberg circle of vertex A). Barycentric equation, center and radius are :*

$$a^2yz + b^2xz + c^2xy - a^2(x+y+z)(y+z) = 0$$

$$N_a \simeq \left(\begin{array}{c} a^2\left(c^2+a^2+b^2\right) \\ \left(a^2+b^2\right)c^2-b^4-a^4 \\ \left(a^2+c^2\right)b^2-c^4-a^4 \end{array}\right) \simeq \left(\begin{array}{c} -a\cos(\omega) \\ b\cos(C+\omega) \\ c\cos(B+\omega) \end{array}\right)$$

$$\rho_A = \frac{a}{2}\sqrt{\cot^2\omega - 3}$$

Proof. Straightforward computation, replacing b^2 by $|A'C|^2$ etc. The form given proves the circular shape. □

Proposition 5.7.6. Tarry point. *Triangle $N_aN_bN_c$ is perspective with ABC and perspector is the Tarry point X(98)=*

$$\frac{1}{a^2b^2+a^2c^2-b^4-c^4} : \frac{1}{b^2c^2+b^2a^2-c^4-a^4} : \frac{1}{c^2a^2+c^2b^2-a^4+b^4}$$

Conversely, Neuberg center N_a is common point of the A cevian of X_{98} and the perpendicular bisector of side BC, etc.

Proof. Direct computation. □

Proposition 5.7.7. *The Steiner angles $\omega_1 > \omega_2$ are defined as follows. $2\omega_1$ is the maximal value of A when ω is given and $2\omega_2$ is the minimal value. We have the following relations :*

$$\begin{gathered} \cot 2\omega_j + 2/\cot\omega_j = \cot\omega \\ \cot\omega_1 = \cot\omega - \sqrt{\cot^2\omega - 3} \\ \cot\omega_2 = \cot\omega + \sqrt{\cot^2\omega - 3} \\ \sin(2\omega_j + \omega) = 2\sin\omega \\ \omega + \omega_1 + \omega_2 = \pi/2 \end{gathered}$$

Proof. First formula comes from (5.21) (at extremum, triangle ABC is isosceles). This gives a second degree equation whose discriminant $\cot^2\omega - 3$ is non negative, and last formula comes from $\cot(\omega_1+\omega_2)$ depends on sum and product of the $\cot\omega_j$. □

Proposition 5.7.8. *Any Neuberg circle is viewed from another vertex under angle 2ϑ where :*

$$\cos\vartheta = 2\sin\omega = \sin(2\omega_j + \omega)$$

Proof. The polar of B cuts circle N_a in two points T_1, T_2 (equation of second degree, $\Delta = a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2$). And we have $2\vartheta = \angle(BT_1, BT_2)$. A better choice is $T_0 = midpoint(T_1, T_2)$ and $\vartheta = \angle(BT_1, BT_0)$ ... taking orientation into account ! □

Results about Brocard angle and Kiepert RH have moved to Proposition 10.18.1.

5.8 Orthogonal projector onto a line

Proposition 5.8.1. *The matrix π_Δ of the **orthogonal projector** onto line $\Delta \simeq [p,q,r]$ –not the line at infinity – is given by :*

$$\pi_\Delta = \Delta \,.\, \boxed{\mathcal{M}} \,.\, {}^t\Delta - \boxed{\mathcal{M}} \,.\, {}^t\Delta \,.\, \Delta \tag{5.22}$$

where matrix $\boxed{\mathcal{M}}$ is defined by (5.17).

Proof. Compute in $\mathbb{R}^3$, and define $W = \boxed{\mathcal{M}} \,.\, {}^t\Delta$ (the projective point associated to W is the orthodir of Δ). In this context, rank one matrix $\boxed{M} \doteq W \,.\, \Delta = \boxed{\mathcal{M}} \,.\, {}^t\Delta \,.\, \Delta$ is interesting : its action can be interpreted as redirecting the output of the standard projector $1/\left(p^2+q^2+r^2\right) {}^t\Delta \,.\, \Delta$. Multiplying $\boxed{M}$ by any point of Δ gives 0, and multiplying by W gives λW where $\lambda = \Delta \,.\, W \neq 0$ (since $\Delta \neq \mathcal{L}_\infty$). Therefore, $(1/\lambda)\boxed{M}$ is a rank one projector of $\mathbb{R}^3$ and $\lambda - \boxed{M}$ is a rank 2 projector of the barycentric plane, the required π_P. □

Proposition 5.8.2. *The matrix σ_Δ of the **orthogonal reflection** wrt line $\Delta \simeq [p,q,r]$ –not the line at infinity – is given by :*

$$\sigma_\Delta = \Delta \,.\, \boxed{\mathcal{M}} \,.\, {}^t\Delta - 2\boxed{\mathcal{M}} \,.\, {}^t\Delta \,.\, \Delta \tag{5.23}$$

where matrix $\boxed{\mathcal{M}}$ is defined by (5.17).

Proof. Obvious from the preceding proof. □

Proposition 5.8.3. Cosine of a projection. *Consider line $\Delta_1 \simeq [p,q,r]$ and use $\overrightarrow{e_1} = (q-r, r-p, p-q)$ as unit vector for this direction. Consider also line $\Delta_2 \simeq [u,v,w]$ and use $\overrightarrow{e_2} = (v-w, w-u, u-v)$ as unit vector for that other direction. Then orthogonal projection π onto Δ_1 transforms Δ_2-vectors into Δ_1-vectors according to :*

$$\pi\left(\overrightarrow{e_2}\right) = \overrightarrow{e_1}\, \frac{\Delta_1 \,.\, \boxed{\mathcal{M}} \,.\, {}^t\Delta_2}{\Delta_1 \,.\, \boxed{\mathcal{M}} \,.\, {}^t\Delta_1}$$

Proof. Formula is homogeneous, as it should be. Vectors $\overrightarrow{e_i}$ are not normalized, that the reason why this formula is square-root free. Formula $2S\boxed{\mathcal{M}} = {}^t\boxed{W} \,.\, \boxed{Pyth} \,.\, \boxed{W}$ indicates that scaling factor can be interpreted in terms of a "cosine of projection". □

Proposition 5.8.4. *Consider the **homothecy** of center $P = p:q:r$ (not on the infinity line) and ratio k (not 0 !). Then points U are transformed as $U \mapsto h(P,k)\,.\,U$ while lines Δ are transformed as $\Delta \mapsto \Delta\,.\,h(P,1/k)$ where :*

$$h(P,k) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] + \frac{1-k}{k\,(p+q+r)} \left[\begin{array}{ccc} p & p & p \\ q & q & q \\ r & r & r \end{array}\right] \tag{5.24}$$

Proof. We want the affine relation $(X-P) = k\,(U-P)$. Expressed in barycentrics, this gives :

$$\left(\begin{array}{c} x \\ y \\ z \end{array}\right) \simeq \left(\begin{array}{c} u \\ v \\ w \end{array}\right) + \frac{(1-k)\,(u+v+w)}{k\,(p+q+r)} \left(\begin{array}{c} p \\ q \\ r \end{array}\right)$$

leading to matrix formula (5.24). Property concerning lines is obvious... another reason to describe points by columns, and lines by rows. It can be seen that eigencolumns of matrix :

$$\pi_0 = \left[\begin{array}{ccc} p & p & p \\ q & q & q \\ r & r & r \end{array}\right]$$

are P and all the points at infinity (straightforward from the definition of h). By duality, all lines through P as well as $\mathcal{L}_\infty$ are the fixed lines of h. □

Chapter 6

Pedal stuff

In a previous life, this Section was intended as foreword to Chapter 5 (orthogonality). Now, this Section is rather a the symmetric aisle of the former Chapter 3 (cevian stuff).

6.1 Pedal triangle

Definition 6.1.1. The **pedal triangle** of point P is the triangle whose vertices are the orthogonal projections of P on the sides of the triangle.

Remark 6.1.2. Crossover the Channel, the pedal triangle is called "triangle podaire", while "pédal triangle" is used to denote the Cevian triangle. Plaisante vérité qu'une rivière borne.

Proposition 6.1.3. *We have the following barycentrics (each point is a column) :*

$$pedal\left(\begin{array}{c} p \\ q \\ r \end{array}\right)=\left[\begin{array}{ccc} 0 & S_c\, q+b^2 p & S_b\, r+c^2 p \\ S_c\, p+a^2 q & 0 & S_a\, r+c^2 q \\ S_b\, p+a^2 r & S_a\, q+b^2 r & 0 \end{array}\right] \tag{6.1}$$
$$where \quad S_a=\left(b^2+c^2-a^2\right)/2, \quad etc.$$

Proof. Use (5.22) and obtain directly the result. □

Proposition 6.1.4. *Condition for an inscribed triangle $P_1P_2P_3$ to be the pedal triangle of some P is :*

$$\frac{q_1-r_1}{q_1+r_1}\,a^2+\frac{r_2-p_2}{p_2+r_2}\,b^2+\frac{p_3-q_3}{p_3+q_3}\,c^2 \tag{6.2}$$

In such a case, point P is given by either following expressions :

$$\left[\begin{array}{c} b^2c^2p_2p_3-\left(S_cr_2-S_ap_2\right)\left(S_bq_3-S_ap_3\right) \\ b^2c^2p_2q_3+\left(S_bq_3-S_ap_3\right)b^2r_2 \\ b^2c^2p_3r_2+\left(S_cr_2-S_ap_2\right)c^2q_3 \end{array}\right]$$

$$\left[\begin{array}{c} a^2c^2p_3q_1+\left(S_ap_3-S_bq_3\right)a^2r_1 \\ a^2c^2q_1q_3-\left(S_ap_3-S_bq_3\right)\left(S_cr_1-S_bq_1\right) \\ a^2c^2q_3r_1+\left(S_cr_1-S_bq_1\right)c^2p_3 \end{array}\right]$$

$$\left[\begin{array}{c} a^2b^2p_2r_1+\left(S_ap_2-S_cr_2\right)a^2q_1 \\ a^2b^2q_1r_2+\left(S_bq_1-S_cr_1\right)b^2p_2 \\ a^2b^2r_1r_2-\left(S_bq_1-S_cr_1\right)\left(S_ap_2-S_cr_2\right) \end{array}\right]$$

Proof. P is on the line through P_1 and orthopoint of BC etc. The required condition is the determinant of these three lines. The various ways of writing P are the wedge product two at a time. A more symmetrical formula would be great... □

6.2 Isogonal conjugacy

Definition 6.2.1. isogonal conjugate. Suppose $P = p : q : r$ is a point not on a sideline of ABC. Let $L(A)$ be the line obtained by reflecting line AP in the internal bisector of angle A. Define $L(B)$ and $L(B)$ cyclically. The lines $L(A)$, $L(B)$, $L(C)$ concur in the isogonal conjugate of P, which has barycentrics :

$$isog\,(x : y : z) = \frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z} \tag{6.3}$$

Remark 6.2.2. The isogonal conjugate of a point P is often noted P^{-1} in ETC since its trilinears are $1/p : 1/q : 1/r$ when those of P are $p : q : r$.

6.3 Cyclopedal conjugate

Proposition 6.3.1 (Matthieu)**.** *When P and U are isogonal conjugates, their pedal triangles are cocyclic and the center of this circle is the middle of P and U (cf Figure 6.1)*

Proof. Straightforward computation. □

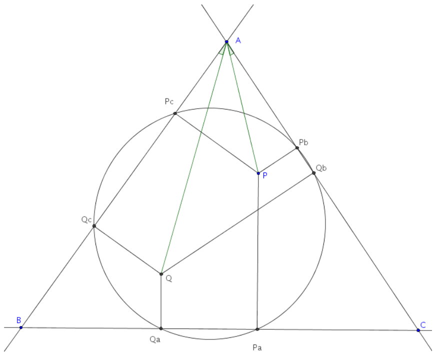


Figure 6.1: Cyclopedal congugates are isogonal conjugates

Remark 6.3.2. The definition of "cyclopedal conjugacy" has been coined to enforce symmetry with the cyclocevian conjugacy, cf. Section 10.19. Some examples are :

point	code	bary	cycp	circumcenter
incenter	$X(1)$	a	$X(1)$	$X(1)$
centroid	$X(2)$	1	$X(6)$	$X(597)$
Lemoine	$X(6)$	a^2	$X(2)$	$X(597)$
circumcenter	$X(3)$	$a^2\left(-a^2+b^2+c^2\right)$	$X(4)$	$X(5)$
orthocenter	$X(4)$	$1/\left(-a^2+b^2+c^2\right)$	$X(3)$	$X(5)$

Chapter 7

Orthogonal stuff

7.1 Steiner line

Proposition 7.1.1 (Simson line)**.** *The pedal vertices of a point U are collinear if and only if the point is on the circumcircle or on the line at infinity. When it exists, this line is called the Simson line of U. When $U \in \mathcal{L}_\infty$, $Simson\,(U) = \mathcal{L}_\infty$. Otherwise, using parameterization (5.15), the tripole of $Simson\,(U)$ is given by :*

$$\left[\begin{array}{c} \dfrac{-a^2\rho + \left(a^2+b^2-c^2\right)\sigma + \left(c^2+a^2-b^2\right)\tau}{\sigma-\tau} \\ \dfrac{\left(a^2+b^2-c^2\right)\rho - 2\,b^2\sigma + \left(b^2+c^2-a^2\right)\tau}{\tau-\rho} \\ \dfrac{\left(c^2+a^2-b^2\right)\rho + \left(b^2+c^2-a^2\right)\sigma - 2\,c^2\tau}{\rho-\sigma} \end{array}\right] \tag{7.1}$$

Proof. The determinant of (6.1) factors into :

$$32\,\left(p+q+r\right)S^2\left(a^2qr+b^2rp+c^2pq\right)$$

The case $p+q+r=0$ is the line at infinity and is not to be discarded, since the orthogonal projection of a point at infinity onto a line at finite distance is the infinity point of this line. In this case, the tripole of $Simson\,(U)$ is the centroid of the triangle. Otherwise, wedge product of two columns of (6.1) leads to the result. □

Proposition 7.1.2 (Steiner line)**.** *Let U be a point in the barycentric plane, define A' as its symmetric relative to side BC and define B', C' cyclically. Then A', B', C' are aligned if and only if point U is either on the circumcircle or on the line at infinity.*
When $U \in \mathcal{L}_\infty$, $A'=B'=C'=U$ and $Steiner\,(U) = \mathcal{L}_\infty$.
When $U = u:v:w \in \Gamma$, so that $Q = isogon\,(U) \in \mathcal{L}_\infty$, then $Steiner\,(U)$ goes through the orthocenter X_4 and the coefficients of $Steiner\,(U)$ are given by :

$$\begin{aligned} Steiner\,(U) &\simeq {}^t isogon\,(U *_b X_4) \simeq {}^t(Q \div_b X_4) \\ &\simeq \left[S_a\,(\tau-\sigma)\,, S_b\,(\rho-\tau)\,, S_c\,(\sigma-\rho)\right] \\ &\simeq \left[\frac{a^2\,S_a}{u}, \frac{b^2\,S_b}{v}, \frac{c^2\,S_c}{w}\right] \end{aligned} \tag{7.2}$$

Proof. Collinearity comes from the Simson line. Then equation of the Simson line (7.1) is transformed using $h\,(U,1/2)$ of (5.24) together with parameterization (5.15). □

Proposition 7.1.3. *When U_1 and U_2 are on the circumcircle, then :*

$$\left(\overbrace{Steiner\,(U_1)\,, Steiner\,(U_2)}\right) = -\frac{1}{2}\left(\overrightarrow{X_3U_1}, \overrightarrow{X_3U_2}\right)$$

where the "overbrace" denotes the oriented angle between two straight lines. Therefore, it exists a one-to-one correspondence between lines through X_4 and points U on the circumcircle. Moreover, Steiner lines relative to diametrically opposed points on the circumcircle are orthogonal to each other.

$$\begin{tikzcd}
S_2 = {}^t(Q_2 \div_b X_4) \arrow[d, "\infty"'] & U_2 \in \Gamma \arrow[l] \arrow[r, dash, "reflection\ in\ X_3"] & U_1 \in \Gamma \arrow[r] & S_1 = {}^t(Q_1 \div_b X_4) \arrow[d, "\infty"] \\
Q_1 = \mathcal{L}_\infty \cap S_2 \arrow[rrr, dashed, "orthopoint"'] \arrow[urr, dotted, "isog"] & & & Q_2 = \mathcal{L}_\infty \cap S_1 \arrow[ull, dotted, "isog"']
\end{tikzcd}$$

$$\begin{aligned}
orthopoint\,(Q) &= \mathcal{L}_\infty \wedge {}^t(Q \div_b X_4) && (7.3)\\
Steiner\,(P) &= X_4 \wedge orthopoint\,(isogon\,(P)) && (7.4)
\end{aligned}$$

Figure 7.1: The orthopoint transform

Proof. From elementary Euclidean geometry... or using $\tan(2\vartheta) = 2\tan\vartheta / \left(1 - \tan^2\vartheta\right)$ and (5.19). □

Corollary 7.1.4. *For each point on the circumcircle, the isogonal conjugate is the orthopoint of the Steiner line (cf. Figure 7.1, and also –far below– Figure 15.4).*

Proof. Let Q_1 be a point at infinity. Take the isogonal conjugate of Q_1 and obtain $U_1 \in \Gamma$. Take the Steiner line of U_1 and obtain S_1. Take the point at infinity of S_1 and obtain Q_2. Take the isogonal conjugate of Q_2 and obtain $U_2 \in \Gamma$. Take the Steiner line of U_2 and obtain S_2. Now Q_1 is the point at infinity of S_2 while Q_1, Q_2 are orthopoints of each other and U_1, U_2 are antipodes of each other on the circumcircle. □

Proposition 7.1.5. *The intersections of the U-Simson line and the nine points circle γ are (i) the midpoint M of UX_4 (ii) the intersection L of $Simson\,(U)$ and $Simson\,(U')$ where U' is the Γ-antipode of U.*

Proof. By definition, $Steiner\,(U)$ is obtained from $Simson\,(U)$ by $h\,(U, 1/2)$. Since $X_4 \in Steiner\,(U)$, point M is on $Simson\,(U)$. Using now $h\,(X_4, 1/2)$, $U \in \Gamma$ becomes $M \in \gamma$ and (i) is proved. Therefore midpoint M' of $U'X_4$ is the γ-antipode of M, while $LM \perp LM'$. □

Proposition 7.1.6. *When Q_1, Q_2 are orthopoints, their barycentric product lies on the orthic axis, i.e. the tripolar of X_4.*

Proof. Use parameterization (5.14), eliminate k, σ in $k\,X = Q_1 *_b Q_2$ and obtain an equation of first degree. □

Example 7.1.7. The following list gives the triples (I, J, K) where $X\,(I)$ and $X\,(J)$ are named orthopoints of each other and $X\,(K)$ is their (named) barycentric product :

30	523	1637
511	512	2491
513	517	3310
514	516	676
2574	2575	647
3307	3308	3310

7.2 Orthopole

Definition 7.2.1. Orthopole. Given a line $\Delta \neq \mathcal{L}_\infty$, define A' as the orthogonal projection of A onto Δ. Then draw the line through A' orthogonally to BC. Acting cyclically leads to three lines that concur in what is called the orthopole of Δ (named O in Figure 7.2). Some orthopoles are given in Table 1.1.

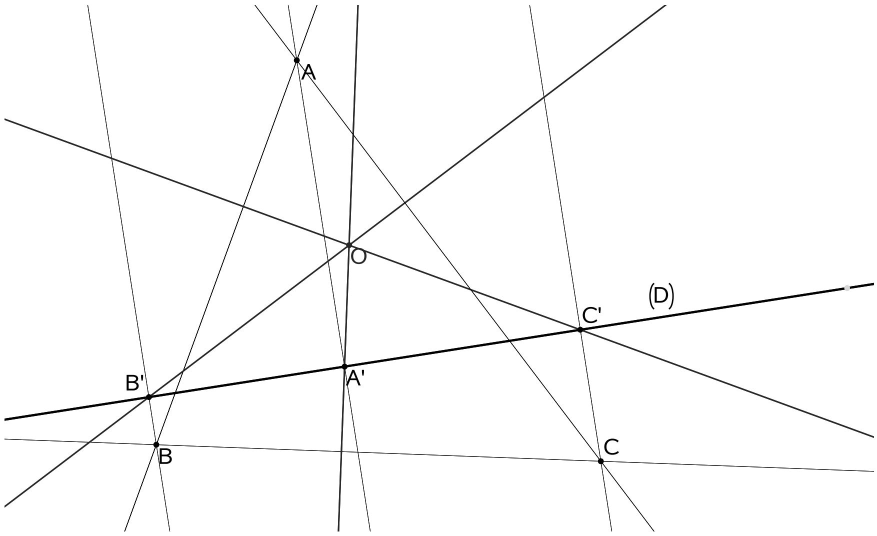


Figure 7.2: Point U is the orthopole of line $A'B'C'$

Proposition 7.2.2. *The barycentrics coordinates of the orthopole of line Δ are given by :*

$$orthopole\,(\Delta) \simeq {}^t(\Delta\,.\,\mathcal{M}) *_b {}^t(\Delta\,.\,\mathcal{N}) \tag{7.5}$$

where $\mathcal{M}$ is defined by (5.17) and $\mathcal{N}$ by :

$$\mathcal{N} = \frac{1}{4S^2}\begin{pmatrix} -S_bS_c & a^2S_a & a^2S_a \\ b^2S_b & -S_cS_a & b^2S_b \\ c^2S_c & c^2S_c & -S_aS_b \end{pmatrix}$$

Proof. Direct computations using the projector formula (5.22) □

Remark 7.2.3. Matrix $\mathcal{N}$ is normalized in order to have eigenvalues $+1$ (once) and -1 (twice). Eigenform relative to $+1$ is $[1,1,1]$, i.e. $\mathcal{L}_\infty$while eigenvector relative to the same $+1$ is X_3, the circumcenter. Therefore, the eigenforms relative to -1 are all the lines through the circumcenter, and the eigenvectors relative to the same -1 are all the points at infinity. This matrix acts like an hyperplane symmetry... before the projective mapping.

Theorem 7.2.4. *Given a line, not $\mathcal{L}_\infty$, the perpendicular through the orthopole is the unique member of the orthogonal pencil that is also a Simson line.*

Proof. Start from Δ_1 and take the point at infinity, obtaining Q_1. Then follow the Steiner movie (Corollary 7.1.4). The required line is the Simson line of point U_1 (there is only one Simson line per direction). It remains to check that orthopole is here. The Steiner line is through $\sigma(\rho-\sigma)S_c+\tau(\rho-\tau)S_b$ etc. □

Proposition 7.2.5. *When points U_1 and U_2 are antipodes on the circumcircle, the orthopole of $Simson\,(U_1)$ is the intersection of $Simson\,(U_2)$ with $Steiner\,(U_1)$.*

Proof. Easily obtained using parameterization (5.15). □

Proposition 7.2.6. *When a line goes through the circumcenter, its orthopole belongs to the nine points circle.*

Proof. Use $\Delta = X_3 \wedge P$, compute the orthopole and substitute. □

Proposition 7.2.7. *When X moves on a line through the centroid X_2, then orthopole of $trilipo\,(X)$ moves on a line through the orthocenter X_4. For example :*

X on line	*orthopole (trilipo (X)) on line*
L(2,1)	*L(4,9)*
L(2,3)	*L(4,6)*
L(2,6)	*L(4,3)=L(2,3)*
L(2,7)	*L(1,4)*

Proof. Suppose X is not X_2 and does not lie on a sideline of triangle ABC. Then, using barycentrics , we have :

$$orthopole(trilipo(X)) \wedge X_4 = \\ S_a\,(y-z) : S_b\,(z-x) : S_c\,(x-z) = (S_a : S_b : S_c) *_b (X \wedge X_2)$$

∎

7.3 Orthojoin

Definition 7.3.1. Orthojoin. Using a general formulation, the orthojoin of a point X, different from X_6, is the orthopole of the tripolar of the isogonal conjugate of X. In other words :

$$\begin{aligned} orthojoin\,(X) &= orthopole\,(X *_b X_{76}) \\ orthojoin\,(isogon\,(X)) &= orthopole\,(tripolar\,(X)) \end{aligned}$$

Remark 7.3.2. This name was coined on the basis that, when using trilinears, $orthojoin\,(X) = orthopole\,({}^tX)$. It is not clear if this concept is really useful.

Proposition 7.3.3. *When $orthojoin\,(X)$ is on the nine-points circle, then X belongs either to line $L(230, 231) = tripolar\,(X_4)$ or to a not so simple cubic (with cubic terms). Examples:*

$X \in tripolar\,(X_4)$	230	231	232	468	523	647	650	27 named
$orthojoin\,(X)$	114	128	132	1560	115	125	11	17 named

Proof. Direct computation. For the circumcircle, the corresponding equation doesn't split. ∎

$$\begin{tikzcd}
Q_{113} \in (\gamma) \arrow[dr, dash, "comcon"'] \arrow[ddd, dash, "\sigma_5"'] & & \widehat{P_{74}} \in (\Gamma) \arrow[ll, dotted, "complem"'] \arrow[dl, dash, "isogon"] \arrow[ddd, dash, "\sigma_3"] \\
& U_{30} \in \mathcal{L}_\infty \arrow[d, dash, "\perp"'] & \\
& \widehat{U_{523}} \in \mathcal{L}_\infty \arrow[dl, dash, "comcon"'] \arrow[dr, dash, "isogon"] & \\
\widehat{Q_{125}} \in (\gamma) & & P_{110} \in (\Gamma) \arrow[ll, dotted, "complem"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Q_{114} \in (\gamma) \arrow[dr, dash, "comcon"'] \arrow[ddd, dash, "\sigma_5"'] & & \widehat{P_{98}} \in (\Gamma) \arrow[ll, dotted, "complem"'] \arrow[dl, dash, "isogon"] \arrow[ddd, dash, "\sigma_3"] \\
& U_{511} \in \mathcal{L}_\infty \arrow[d, dash, "\perp"'] & \\
& \widehat{U_{512}} \in \mathcal{L}_\infty \arrow[dl, dash, "comcon"'] \arrow[dr, dash, "isogon"] & \\
\widehat{Q_{115}} \in (\gamma) & & P_{99} \in (\Gamma) \arrow[ll, dotted, "complem"]
\end{tikzcd}$$

Figure 7.3: The orthojoin duality

Proposition 7.3.4. *Let U be on the line at infinity, $\widehat{U} = orthopoint\,(U)$ and $Q = comcon\,(U)$. Then $\widehat{U}$ is also on $\mathcal{L}_\infty$, while Q is on the nine points circle γ. The locus of points X such that $orthojoin\,(X) \in QU$ is the union of two lines, whose tripolar are $\widehat{U}$ and $P = isogon\left(\widehat{U}\right)$ (P is on the circumcircle Γ). Figure 7.3 describes how these points are related in two configurations. More examples (sorted on U) are given in Table 7.1.*

P	X on $tripolar\,(P)$	$\widehat{U}$	X on $tripolar\left(\widehat{U}\right)$	$orthojoin\,(X)$ (on QU)		Q,U	
110	**3**,15,*511*,...			1514,1524,*1561*...		113	30
		523	115,125,*690*...	1553,1554,*2686*...		113	30
99	**2**,230			1513,114		114	511
		512	2491,3121...	2679,2683,...		114	511
98	**523**, 1640			*115*, 2682		115	512
		511	2491,3569	2679,2682		115	512
104	**650**			*11*		*11*	513
		517	3310	3259		11	513
109	**41**,56,73			1535,1542,1549		117	515
		522	11,2170	1521,???		117	515
101	**31**,42,55			1530,1536,1541...		118	516
		514	11,244,676	1521,???,1566		118	516
100	**1**,9,37,44			1512,1519,1532,1537		119	517
		513	244,2170,3121	???,???,2683		119	517
74	**647**, 2433			*125*, 3154		125	523
		30	1637	3258		125	523

tripolar(P) is the line from **bold** to X(6)

Table 7.1: Some orthojoin configurations

7.4 Rigby points

Definition 7.4.1. Rigby points. When points U_1, U_2 are on the circumcircle, the Rigby-Simson point –aka $RS\,(U_1, U_2)$– is defined as the intersection of the Simson lines of both points, while the Simson-Rigby point –aka $SR\,(U_1, U_2)$– is the unique $U_3 \in \Gamma$ whose Simson line is orthogonal to U_1U_2. These points are discussed Honsberger (1995, Chapter 11: The Orthopole, pages 124-136).

Proposition 7.4.2. *The Rigby-Simson point K is the orthopole of each sideline of triangle $U_1U_2U_3$, while U_3 is the intersection of $U_1 isog\,(U_2)$ with $U_2 isog\,(U_1)$. In this document, the following expressions will be used :*

$$RS\,(P,U) \quad = \quad orthopole\,(P \wedge U) \tag{7.6}$$

$$\begin{aligned} SR\,(P,U) &= isog\,((P \wedge U) \wedge \mathcal{L}_\infty) \\ &= \frac{a^2}{(q+r)\,u-(v+w)\,p} : \frac{b^2}{(p+r)\,v-(u+w)\,q} : \frac{c^2}{(p+q)\,w-(u+v)\,r} \end{aligned} \tag{7.7}$$

Proof. By using parameterization (5.15) into relations :

$$\begin{aligned} orthopole\,(P \wedge U) &\simeq Simson\,(P) \wedge Simson\,(U) \\ isog\,((P \wedge U) \wedge \mathcal{L}_\infty) &\simeq (P \wedge U^*) \wedge (U \wedge P^*) \quad \square \end{aligned}$$

Remark 7.4.3. Many expressions can be given for $RS\,(P,U)$ and $SR\,(P,U)$, since they only make sense when $P, U \in \Gamma$. Equivalence has to be tested under parameterization (5.15). Among them,

we have the Peter Moses (2004/10) expression (converted to barycentrics) :

$$\begin{array}{rcl} K & \simeq & \dfrac{b^2c^2}{a}\left(\begin{array}{c} \dfrac{1}{r^2v^2\left(\frac{q\cos(A)}{b}-\frac{p\cos(B)}{a}\right)\left(\frac{w\cos(A)}{c}-\frac{u\cos(C)}{a}\right)} \quad \cdots \\ -\dfrac{1}{q^2w^2\left(\frac{r\cos(A)}{c}-\frac{p\cos(C)}{a}\right)\left(\frac{v\cos(A)}{b}-\frac{u\cos(B)}{a}\right)} \end{array}\right) \\ U_3 & \simeq & \dfrac{qw-rv}{rwb^2-qvc^2}:\dfrac{ru-pw}{c^2up-a^2wr}:\dfrac{pv-uq}{a^2qv-b^2pu} \end{array}$$

Example 7.4.4. Centers X(2677) to X(2770) are examples of Rigby-Simson points and Simson-Rigby points. In the following table, the first three of a quadruple is a (sorted) triangle $U_1U_2U_3$ and its Rigby point.

74	98	691	?	99	1380	1380	?	102	104	2222	?
74	99	842	?	99	2378	2379	?	103	104	1308	?
74	110	477	3258	100	101	1308	?	104	840	1292	?
74	1113	1114	125	100	104	953	3259	104	1381	1382	11
74	1294	1304	?	100	105	840	?	107	110	1304	?
98	110	842	2682	100	109	2222	?	110	110	476	1553
98	843	1296	?	100	110	1290	?	110	112	935	1554
98	1379	1380	115	100	1381	1381	?	110	827	1287	?
99	110	691	?	100	1382	1382	?	110	930	1291	?
99	111	843	?	101	109	929	1521	110	1113	1113	?
99	1379	1379	?	102	103	929	?	110	1114	1114	?

Proposition 7.4.5. ***Third point.*** *For each point U on the circumcircle, it exists exactly one other point U_2 such that $SR(U,U_2)=U$. This point is the other intersection of Γ with UU^* (the line through U and orthogonal to Simson(U) and is given by third(U) =*

$$\frac{a^2}{u^2(bw+cv)(bw-cv)}:\frac{b^2}{v^2(cu+aw)(cu-aw)}:\frac{c^2}{w^2(av+bu)(av-bu)}$$

Proof. When solving $SR(U,X)=U$ with SR as given in (7.7), condition $U\in\Gamma$ appears in the elimination process. Using therefore parameterization (5.15) and eliminating k, p in $SR(U,p:q:r)=kU$, the result obtained in p is equivalent to membership of X to a line that is easily found and proven to be UU^*. Then finding X as $\Gamma\cap UU^*$ is easy since the equation splits (the first intersection is obviously U itself). □

Remark 7.4.6. Relation $SR(U,\mathit{third}(U))=U$ is not granted for a random point U in the triangle plane. But this relation obviously holds when restricting U to the circumcircle.

Proposition 7.4.7. ***Simson-Moses point.*** *If points U_1,U_2 are on the circumcircle then, using isoconjugacy wrt pole $P=X_6$ (isogonal conjugacy), the intersection of lines $U_1(U_2)^*_P$ and $U_2(U_1)^*_P$ is point $SR(U_1,U_2)$. When another isoconjugacy is used, the intersection remains on the circumcircle. Its barycentrics are :*

$$S^*_P(U_1,U_2)=\frac{-w_1v_2+v_1w_2}{qw_1w_2-rv_1v_2}:\frac{-w_1u_2+u_1w_2}{pw_1w_2-ru_1u_2}:\frac{-v_1u_2+u_1v_2}{pv_1v_2-qu_1u_2}$$

Proof. The barycentrics are straightforward, while parameterization (5.15) leads to the other properties. □

Definition 7.4.8. In ETC, the Simson-Moses point is computed using $P=X_2$ (isotomic conjugacy) in $S^*_P(U_1,U_2)$, and noted $SM(U_1,U_2)$. Centers X(2855) to X(2868) are examples of Simson-Moses points.

7.5 Orthology

Definition 7.5.1. Two triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ are orthologic when perpendiculars from the vertices of $\mathcal{T}_1$ to the corresponding sides of $\mathcal{T}_2$ are concurrent.

Proposition 7.5.2. *Orthology is a symmetric relation.*

Proof. Using computer, symmetry is straightforward : condition of concurrence is the product of $\det \mathcal{T}_2$ by a polynomial that is invariant by exchange of the two triangles. □

Example 7.5.3. Cevian triangles of X(2) and X(7) are orthologic. Orthology centers are X(10) and X(1).

7.6 Orthocorrespondents

Definition 7.6.1. Suppose P is a point in the plane of triangle ABC. The perpendiculars through P to the lines AP, BP, CP meet the lines BC, CA, AB, respectively, in collinear points. Let L denote their line. The trilinear pole of L is $P^{\perp}$, the *orthocorrespondent of* P. This definition is introduced in Gibert (2003). If $P = p : q : r$ is given in barycentrics, then $P^{\perp} = u : v : w$ is given by :

$$-\left(b^2+c^2-a^2\right)p^2+\left(a^2-b^2+c^2\right)pq+\left(a^2+b^2-c^2\right)pr+2qra^2$$

Remark 7.6.2. If follows that if $P = x : y : z$ is given in trilinears, then $P^{\perp}$ has trilinears given cyclically by :

$$yz+(-x\cos A+y\cos B+z\cos C)x$$

Example 7.6.3. Pairs (I,J) for which the orthocorrespondent of X(I) is X(J) include the following:

1	57	11	651	62	2005	109	1813	125	648	1566	677
2	1992	13	13	80	2006	111	895	132	287	1785	57
3	1993	14	14	98	287	112	**110**	186	1994	1845	2006
4	2	15	62	100	1332	113	2986	242	1999	1878	1997
5	1994	16	61	101	1331	114	2987	403	1993	3563	2987
6	1995	19	2000	103	1815	115	**110**	468	1992		
7	1996	32	2001	105	1814	117	2988	915	2990		
8	1997	33	2002	106	1797	118	2989	917	2989		
9	1998	36	2003	107	648	119	2990	1300	2986		
10	1999	61	2004	108	651	120	2991	1560	895		

Proposition 7.6.4. *The orthocorrespondent of every point on the line at infinity is the centroid. Conversely, given a finite point U, different from the centroid, it exists exactly two* orthoassociate points P_1 *and* P_2 *(real or not, distinct or not) that share the same orthocorrespondent U. When P_1 is given, then :*

$$p_2 \simeq (q_1+r_1)p_1+\frac{a^2-b^2+c^2}{a^2-b^2-c^2}\,q_1^2+\frac{a^2+b^2-c^2}{a^2-b^2-c^2}\,r_1^2$$

When U is given, the condition of reality is $\Delta \geq 0$ where :

$$\Delta \doteq S^2(u+v+w)^2-u(w+v)S_c\,S_b-v(u+w)S_a\,S_c-w(u+v)S_b\,S_a$$

$S =$ area and $S_a = \left(b^2+c^2-a^2\right)/2$. Then, cyclically, we have :

$$p_1,\, p_2 \simeq \left(\begin{array}{r} S\left((u-w)(u+v-w)S_b+(u-v)(u-v+w)S_c\right) \\ \pm\left((u-w)S_b+(u-v)S_c\right)\sqrt{\Delta} \end{array}\right)$$

Proof. Write *orthocorr* $(P) = kU$ and eliminate (rationally) k, p. Obtain a second degree equation, whose discriminant is $S^2\,(v-w)^2\,\Delta$. In order to obtain a symmetrical form for the barycentrics $p : q : r$, all these expressions must be simplified using $\left(\sqrt{\Delta}\right)^2 = \Delta$, then rationalized and simplified again, and finally normalized using $p+q+r=1$. □

7.7 Isoscelizer

An isoscelizer is a line perpendicular to an angle bisector. If P is a point, then the A-isoscelizer of P is the line $L(P, A)$ through P perpendicular to the line that bisects vertex angle A; the B- and C- isoscelizers are defined cyclically. Let D and E be the points where $L(P, A)$ meets sidelines AB and AC. Unless $D = E = A$, the triangle ADE is isosceles.

In ETC, there are several triangle centers defined in terms of isoscelizers. These were discovered or invented by Peter Yff, in whose notebooks the word *isoscelizer* dates back to 1963.

7.8 Orion Transform

Definition 7.8.1. Reflect P through the sidelines of its cevian triangle $A_P B_P C_P$. The obtained triangle is perspective with triangle ABC, and the perspector is called the Orion transform of P. In barycentrics, we have :

$$OT1\,(P) = p\left(-\frac{a^2}{p^2} + \frac{b^2}{q^2} + \frac{c^2}{r^2} + 2\,\frac{S_a}{qr}\right) :: \text{ etc}$$

Proposition 7.8.2. *(Ehrmann, 2003) For any point P, the reflection triangle and the anticevian triangle of the same point P are perspective, and the perspector is the isogonal conjugate of P wrt the orthic triangle.*

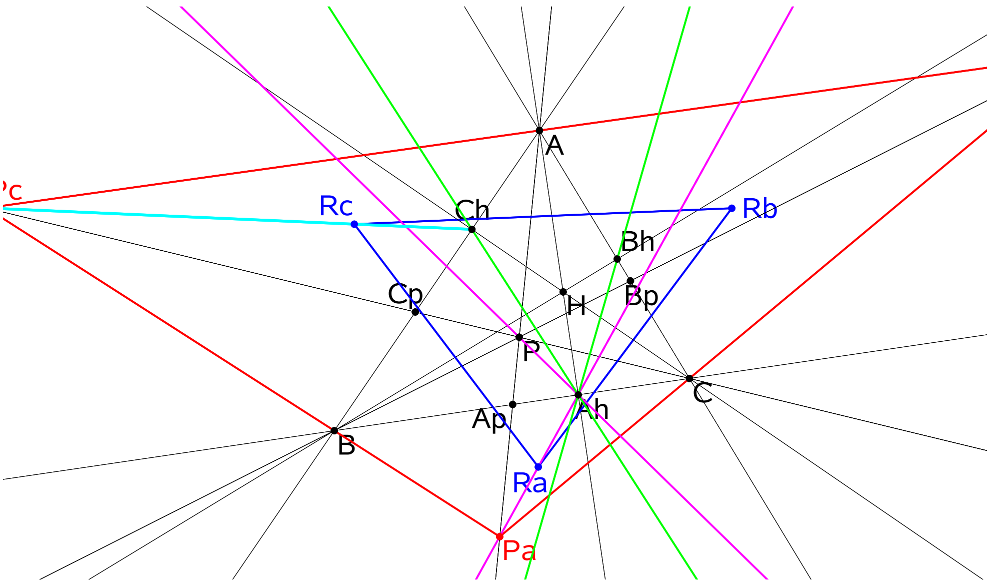


Figure 7.4: Reflection triangle and its perspector.

Proof. Let $P_A P_B P_C$ be the anticevian triangle of P, $A_P B_P C_P$ its cevian triangle and $A_H B_H C_H$ the orthic triangle. Division $(A,\, A_P,\, P,\, P_A)$ is harmonic; so is pencil $(A_H A,\, A_H A_P,\, A_H P,\, A_H P_A)$ where $A_H A \perp A_H A_P$ (don't forget that $A_H A_P = BC$!). Hence :

1. line BC is a bisector of $\angle\,(A_H P,\, A_H P_A)$. Thus R_A, the reflection of P, is on $A_H P_A$. Proving concurrence of lines $P_V R_V$ is now proving concurrence of lines $P_V V_H$ for $V = A, B, C$.

2. line $A_H A$ is a bisector of $\angle\,(A_H P,\, A_H P_A)$. But $A_H A$ is also a bisector of $\angle\,(A_H B_H,\, A_H C_H)$; it follows that $A_H P$ and $A_H P_A$ are isogonal wrt $A_H A_B$ and $A_H C_H$. Since the $V_H P$ obviously concur in P, the $P_V V_H$ concur wrt the isogonal conjugate of P wrt the orthic triangle.

$\square$

Computer-aided proof. . The barycentrics of the reflection triangle are :

$$\mathcal{T}_R = \begin{pmatrix} -a^2 p & 2\,S_c q + b^2 p & 2\,S_b r + c^2 p \\ 2\,S_c p + a^2 q & -b^2 q & 2\,S_a r + c^2 q \\ 2\,S_b p + a^2 r & 2\,S_a q + b^2 r & -c^2 r \end{pmatrix}$$

Perspectivity is obvious and the perspector is :

$$OT2\,(P) = p\,(-S_a p + S_b q + S_c r) : q\,(S_a p - S_b q + S_c r) : r\,(S_a p + S_b q - S_c r)$$

The link with the orthic triangle is given in Figure 17.1. □

Example 7.8.3. Among pairs (I, J) for which $OT(X(I)) = X(J)$ are these:

1	35	4	24	8	2057	20	2060	69	2063	99	249
2	69	6	2056	15	2058	40	2061	75	2064	100	59
3	2055	7	57	16	2059	63	2062	98	2065	110	250

Chapter 8

Circumcevian stuff

8.1 Circum-cevians, circum-anticevians

Definition 8.1.1. Circumcevian. Let P be a point, not on Γ (the circumcircle of ABC). Let A' be the other intersection of line AP with Γ and define B', C' cyclically. Then $A'B'C'$ is the circumcevian triangle of P. Using barycentric columns,

$$circumcevian(P) = \begin{pmatrix} \dfrac{-qra^2}{c^2q+b^2r} & p & p \\ q & \dfrac{-rpb^2}{a^2r+c^2p} & q \\ r & r & \dfrac{-pqc^2}{b^2p+a^2q} \end{pmatrix} \tag{8.1}$$

Proposition 8.1.2. *A circumcevian triangle is a central triangle. When P is on Γ, the corresponding triangle is totally degenerated. Three points on the circumcircle form a circumcevian triangle when the triangle obtained by killing the diagonal of the matrix is a cevian triangle.*

Proof. Centrality follows directly from (8.1) (that's a reason to keep denominators), while killing the diagonal gives the intersections with sidelines. Determinant is $\Gamma(P)^3$ over $\prod a^2q+b^2p$. □

Definition 8.1.3. Circum-anticevian. Consider the anticevian triangle $P_AP_BP_C$ of a point P that is not a vertex of triangle ABC. Line P_BP_C cuts circumcircle Γ at A. Let A' be the other intersection and define B', C' cyclically. Then $A'B'C'$ is the circum-anticevian triangle of P. Using barycentric columns,

$$circumanticevian(P) = \begin{pmatrix} \dfrac{qra^2}{c^2q-b^2r} & -p & p \\ q & \dfrac{rpb^2}{a^2r-c^2p} & -q \\ -r & r & \dfrac{pqc^2}{b^2p-a^2q} \end{pmatrix} \tag{8.2}$$

Proposition 8.1.4. *This triangle should be a central triangle (does the definition allows that ?). When one of the three other points $\pm p : \pm q : \pm r$ is on Γ, the circum-anticevian triangle degenerates.*

8.2 Steinbart transform

Definition 8.2.1. Exceter point. The circumcevian triangle of the centroid, X_2, is perspective to the tangential triangle $\mathcal{A}_6$. The perspector, X_{22}, is named Exeter point, for Phillips Exeter Academy in Exeter, New Hampshire, USA, where X_{22} was detected in 1986 using a computer.

Definition 8.2.2. Steinbart transform. The circumcevian triangle of a point P is ever perspective to the tangential triangle $\mathcal{A}_6$. The corresponding perspector has been called Steinbart point by Funck (2003) and was called in TCCT (p. 201). This transformation carries triangle centers to triangle centers. Using barycentrics :

$$Steinbart\,(P) = a^2\left(\frac{b^4}{q^2}+\frac{c^4}{r^2}-\frac{a^4}{p^2}\right) : b^2\left(\frac{a^4}{p^2}-\frac{b^4}{q^2}+\frac{c^4}{r^2}\right) : c^2\left(\frac{a^4}{p^2}+\frac{b^4}{q^2}-\frac{c^4}{r^2}\right)$$

Example 8.2.3. On the circumcircle, Steinbart transform is the identity. Here is a list of other (I,J) such that Steinbart(X(I)) = X(J) :

1	3	14	1606	56	1616	162	1624	365	55
2	22	17	1607	57	1617	163	1625	366	1631
3	1498	18	1608	58	595	174	1626	509	1486
4	24	19	1609	59	1618	188	2933	648	1632
5	1601	21	1610	63	1619	251	1627	651	1633
6	6	25	1611	64	1620	254	1628	662	1634
7	1602	28	1612	81	1621	259	198		
8	1603	31	1613	83	1078	266	56		
9	1604	54	1614	84	1622	275	1629		
13	1605	55	1615	88	1623	284	1630		

Points X(1601)-X(1634) have been contributed in ETC by Jean-Pierre Ehrmann (August 2003).

Remark 8.2.4. See Grinberg (2003d) and his Extended Steinbart Theorem in Hyacinthos #7984, 2003/09/23.

8.3 Circum-eigentransform

Definition 8.3.1. The circum-eigentransform of point $U = u : v : w$, different from X_6, is the eigencenter of the circumcevian triangle of point U and is denoted by $CET(U)$. In trilinears, we have :

$$\frac{avw}{av^2+aw^2-buv-cuw} : \frac{bwu}{bw^2+bu^2-cvw-avu} : \frac{cuv}{cu^2+cv^2-awu-bwv}$$

and in barycentrics (cyclically) :

$$\frac{a^2vw}{a^2c^2v^2+a^2b^2w^2-b^2c^2uv-bc^2uw}$$

Proposition 8.3.2. *Point $CET(U)$ lies on the circumcircle, and we have $CET(U) = isog\,(U)$ if and only if $U \in \mathcal{L}_\infty$.*

Remark 8.3.3. My own computations are leading to :

$$\frac{a^2vw}{-a^2c^2v^2+a^2b^2w^2-b^2c^2uv+bc^2uw}$$

This point is also on the circumcircle, but property $CET(U) = isog\,(U)$ if and only if $U \in \mathcal{L}_\infty$ is lost. Some signs have changed : why ?

Example 8.3.4. Apart from points on the infinity line, pairs (I, J) such that $X(J) = CET(X(I))$ include :

1	106	41	767	74	1294	238	741
2	729	42	2368	75	701	265	1300
3	1300	43	106	81	2375	670	3222
4	1294	44	106	110	99	694	98
9	1477	55	2369	125	827	895	2374
19	2365	56	2370	184	2367	1084	689
25	2366	57	2371	194	729	1279	1477
31	767	58	2372	213	2368	1634	689
32	2367	67	2373	219	2376		
37	741	69	2374	220	2377		

Exercise 8.3.5. For a given point P on the circumcircle, which points U satisfy $CET(U) = P$? For example, CET carries each of the points X(1), X(43), X(44), X(519) to X(106).

8.4 Dual triangles, DC and CD Points

Definition 8.4.1. Dual triangle. Suppose DEF is a triangle (at finite distance) in the plane of triangle ABC. Let D' be the isogonal conjugate of the point at infinity of line EF. Define E' and F' cyclically. The triangle $D'E'F'$ is called the *dual* of DEF. Its vertices lie on the circumcircle.

Proposition 8.4.2. *The dual triangle $D'E'F'$ characterizes the class of all triangles homothetic to DEF. Moreover, this triangle is similar to the original one.*

Proof. Vertices of $D'E'F'$ depends only on direction of sidelines DE, EF, FD and conversely. Similarity between DEF and $D'E'F'$ can be proved in many ways. Brute force method : Pythagoras Theorem 5.2.4 applied to both triangles leads to proportional sidelengths. □

Remark 8.4.3. (Proof of similarity follows from Theorem 6E in TCCT, as the "gamma triangle" there is the dual of a triangle whose sidelines are respectively perpendicular to those of DEF.)

Definition 8.4.4. DC point. Suppose $U = u : v : w$ is a point having cevian triangle DEF and dual triangle $D'E'F'$. It happens that the later triangle is also the circum-anticevian triangle of some point. This point will be described as $DC\,(U)$. Using barycentrics :

$$DC\,(U) = \frac{1}{u\,(b\,v + c\,w)} : \frac{1}{v\,(a\,u + c\,w)} : \frac{1}{w\,(a\,u + b\,v)}$$

Remark 8.4.5. The barycentrics of triangle $D'E'F'$ are :

$$\begin{array}{ccc} \dfrac{a^2}{wu - vu} & \dfrac{-a^2}{wu + vu} & \dfrac{a^2}{wu + vu} \\ \dfrac{b^2}{vu + wv} & \dfrac{b^2}{vu - wv} & \dfrac{-b^2}{vu + wv} \\ \dfrac{-c^2}{wv + wu} & \dfrac{c^2}{wv + wu} & \dfrac{c^2}{wv - wu} \end{array}$$

Proposition 8.4.6. *To construct $DC(U)$ from U and $D'E'F'$, let $A' = AD' \cap BC$ and let A" be the harmonic conjugate of A' with respect to B and C. Define B" and C" cyclically. The lines AA", BB" and CC" concur in $DC(U)$. We have also the formula :*

$$DC\,(U) = cevamul\,(isog\,(U)\,,\, X\,(6))$$

Example 8.4.7. There are 115 pairs (I,J) such that I<2980 and X(J) = DC(X(I)). Among them (1,81), (2,6), (3,275), (4,2), (5,288), (6,83), (7,1), (8,57), (9,1170), (10,1171). The longest chain for this relation is : 69 ↦ 4 ↦ 2 ↦ 6 ↦ 83 ↦ 3108.

Proposition 8.4.8. *Inversely, the circum-anticevian triangle of a point P is the dual of the cevian triangle of a point CD(P), given for $P = p : q : r$ by the inverse of the DC-mapping; that is:*

$$\frac{1}{-a^2qr + b^2pr + c^2pq} : \frac{1}{a^2qr - b^2pr + c^2pq} : \frac{1}{a^2qr + b^2pr - c^2pq}$$

In other words, we have :

$$CD\,(P) = isog\,(cevadiv(P,\, X(6)))$$

8.5 Saragossa points

Definition 8.5.1. Saragossa points. Let P be a point not on the circumcircle of ABC. Let $\mathcal{T}' = A'B'C'$ be the cevian triangle of P and $\mathcal{T}'' = A'', B'', C''$ the circumcevian triangle of P. Consider triangle $\mathcal{T}$ that is the desmic mate of $\mathcal{T}'$ and $\mathcal{T}''$, i.e. the triangle whose vertices are $U = B'C'' \cap B''C'$, $V = C'A'' \cap C''A'$ and $W = A'B'' \cap A''B'$. Then (Figure 8.1) triangles ABC, $\mathcal{T}'$, $\mathcal{T}''$ and $\mathcal{T}$ are pairwise perspective ((Grinberg, 2003c). The first, second and third Saragossa points of P are the perspectors of $\mathcal{T}$ with, respectively, ABC, $\mathcal{T}'$ and $\mathcal{T}''$. The name *Saragossa* refers to the king who proved Ceva's theorem before Ceva did (Hogendijk, 1995).

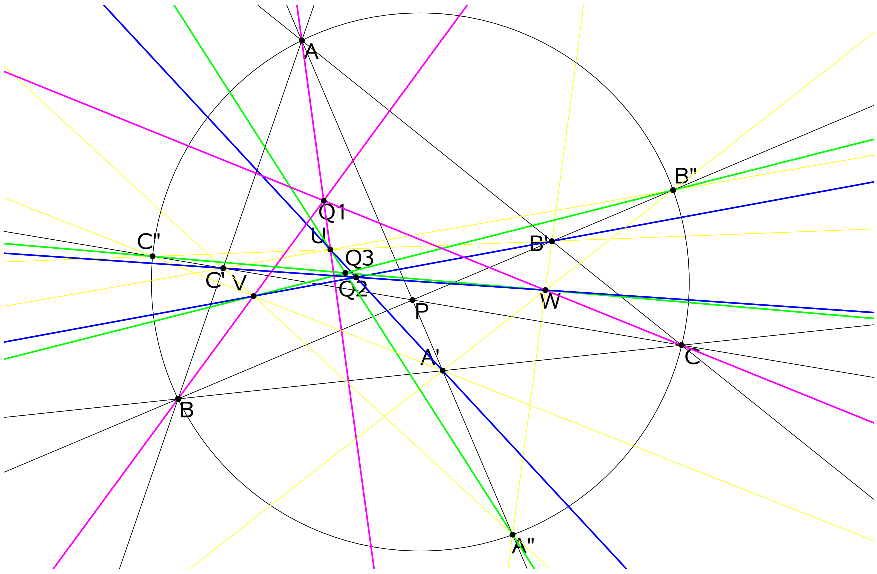


Figure 8.1: Saragossa points of point P

Proposition 8.5.2. *The barycentrics of the Saragossa points of $P = p : q : r$ are (cyclically) :*

$$\begin{aligned} g_1(a,b,c) &= \frac{a^2}{a^2qr - (a^2qr + b^2pr + c^2pq)} \\ g_2(a,b,c) &= p - \left(\frac{1}{c^2q} + \frac{1}{rb^2}\right)\left(a^2qr + b^2pr + c^2pq\right) \\ g_3(a,b,c) &= 2\,p - \left(\frac{1}{c^2q} + \frac{1}{rb^2}\right)\left(a^2qr + b^2pr + c^2pq\right) \end{aligned} \tag{8.3}$$

Points P, Q_2, Q_3 are clearly collinear. When one of the Saragossa points is equal to P then P is $X(6)$ or lies on the circumcircle.

Proof. Computations are straightforward. □

Example 8.5.3. The following table give the Saragossa points of the $X\,(I)$ whose number is given in the first line.

1	2	3	4	5	6	19	21	24	25	28	31
58	251	4	54	1166	6	284	961	847	2	943	81
386	1180	1181	?	?	6	1182	1183	?	1184	?	1185
1193	1194	185	389	?	6	1195	?	?	1196	?	1197

8.6 Vertex associates

Definition 8.6.1. Vertex associate. Consider the circumcevian triangles $A_pB_pC_p$, $A_uB_uC_u$ of points P, U (not both on the circumcircle) and draw their may be degenerate vertex triangle $\mathcal{T}$ i.e. the triangle whose sidelines are A_pA_u, B_pB_u, C_pC_u. It happens that $\mathcal{T}$ is perspective to ABC : the corresponding perspector X is called the vertex associate of P and U.

Proposition 8.6.2. *When $P \in \Gamma$ but $U \notin \Gamma$, $A_pB_pC_p$ and $\mathcal{T}$ are totally degenerate at P, so that $X = P$ (regardless of U). Otherwise, the barycentrics of the vertex associate of $p : q : r$ and u:v:w are (cyclically) :*

$$vertexthird\,(P,U) = \frac{a^2}{wra^4qv - p\,(wb^2 + vc^2)\,u\,(rb^2 + c^2q)}$$

Proof. When both P, U are on Γ, both circumcevians are totally degenerate and X is not defined. □

Remark 8.6.3. The definition of vertex conjugate allows X = U. To extend the geometric interpretation to the case that X = U, as X approaches U, the vertex triangle approaches a limiting triangle which we call the tangential triangle of U, a triangle perspective to ABC with perspector U-vertex conjugate of U.

Proposition 8.6.4. *When P is not on Γ, but U is on P_Γ^* (the Γ-polar of P), then $\mathcal{T}$ is totally degenerate to a point X that is the Γ-pole of line PU. Finally, triangle PUX is autopolar wrt Γ.*

Proof. Concurrence of A_pA_u, B_pB_u, C_pC_u in a point X is straightforward, and $X \in P_\Gamma^*$ too. $\square$

Proposition 8.6.5. *Operation vertexthird is commutative and "formally involutory" i.e. :*

$$vertexthird\,(P,\, vertexthird\,(P,U)) \simeq U$$

unless P lies on the circumcircle (where $vertexthird\,(P,U) = P$, regardless of U).

Proof. Commutativity is from the very definition, and the formal involutory property is from straightforward computations. It remains only to track degeneracies. Determinant of the vertex triangle $\mathcal{T}$ is the square of determinant of the corresponding trigon ($\mathcal{T}$ is either a genuine triangle, or totally degenerate). The factors are the conditions for $U \in \Gamma$, $P \in \Gamma$ and the condition for one point to be of the Γ-polar of the other. $\square$

Exercise 8.6.6. In the general case, what can be said about the way the three circumcevian triangles are lying on Γ ?

Example 8.6.7. Here are some vertex conjugates X(I), X(J), X(K) :

	1	2	3	4	5	6	7	8	9
1	56	3415	84	3417		2163	3418		3420
2	3415	25	3424	3425		1383			
3	84	3424	64	4		3426	3427		
4	3417	3425	4	3		3431			
5					3432				
6	2163	1383	3426	3431		6			
7	3418		3427				3433		
8								3435	
9	3420								1436

Proposition 8.6.8. *For a given U, not on the circumcircle, the associated first Saragossa point (8.3) is the sole and only point X such that $vertexthird\,(U,X) = X$.*

Proof. We want $U = VT\,(X, VT\,(X,U)) = VT\,(X,X)$ while $sarag1\,(VT\,(P,P)) = P$ is straightforward. $\square$

Proposition 8.6.9. *Vertex association wrt X_3 maps the Darboux cubic to the Darboux cubic (X_3 is the reflection center of this cubic, whose pole is X_6 and pivot X_{20}). The appearance of (I,J) in the following list means that X(I),X(J) are on the Darboux cubic and that X(3),X(I),X(J) are vertex associates :*

1	3	4	20	40	1490	1498	2131	3182
84	64	4	3346	3345	3347	3348	3183	3354

Chapter 9

About conics

Notation 9.0.1. $\mathcal{C}$ is a conic, $\boxed{\mathcal{C}}$ is the matrix of the ponctual équation while $\boxed{\mathcal{C}^*}$ is the matrix of the tangential equation. Point P is (often) the perspector, U is (often) the center

9.1 Tangent to a curve

Proposition 9.1.1. *Consider an algebraic curve $\mathcal{C}$ (not necessarily a conic). When the defining polynomial $\mathcal{C}(x, y, z) = 0$ is homogeneous, the tangent to $\mathcal{C}$ at point $P = p : q : r$ is given by :*

$$\overrightarrow{grad}\left(\mathcal{C}\right)_{p:q:r} = \left[\left(\frac{\partial \mathcal{C}}{\partial x}\right)_{X=P}, \left(\frac{\partial \mathcal{C}}{\partial y}\right)_{X=P}, \left(\frac{\partial \mathcal{C}}{\partial z}\right)_{X=P}\right] \tag{9.1}$$

Proof. Let $P \in \mathcal{C}$ be the contact point and $P + kU$ be a point in the vicinity. If we require $P + kU \in \mathcal{C}$, we must have $\mathcal{C}(P + kU) - \mathcal{C}(P) = \mathrm{O}\left(k^2\right)$ and this is $\overrightarrow{grad}\left(\mathcal{C}\right)_{p:q:r} . U = 0$. But polynomial $\mathcal{C}$ is homogeneous and we have $\overrightarrow{grad}\left(\mathcal{C}\right)_{p:q:r} . P = dg\left(\mathcal{C}\right) \mathcal{C}$ and the result follows. □

Exercise 9.1.2. Use parametrization (5.15) to describe the points P of the circumcircle. Obtain the tangent at P. Take the orthodir and obtain the normal. Differentiate and wedge to catch the contact point of the enveloppe of all the normals... and obtain X(3).

Definition 9.1.3. Pole and polar. The polar line of point X with respect to an algebraic (homogeneous) curve $\mathcal{C}$ is the line whose affix is the gradient of $\mathcal{C}$ evaluated at point X. Point X is called a pole of its polar.

Remark 9.1.4. When point X is a simple point on an algebraic curve, its polar is nothing but the line tangent at X to the curve. Finding all the points whose polar is a given line is not an easy task in the general case.

Proof. Well known result. In fact, this is the rationale for the concept of polarity. □

9.2 Folium of Descartes

Folium of Descartes is not a conic ! But, in our opinion, it could be useful to see how some general methods are working in the general case, before to use them in the rather specific situation of the algebraic curves of degree two.

Definition 9.2.1. The **folium** $\mathcal{F}$ is the curve used by Descartes to check his methods regarding the coordinate system. The cartesian equation of this curve is $x^3 + y^3 - 6xy = 0$, and its homogeneous equation is

$$X^3 + Y^3 - 6\, X\, Y\, T = 0$$

Proposition 9.2.2. *The folium presents a double point at $0 : 0 : 1$. If we cut by the line $Y = pX$, we obtain the parameterization : $M \simeq 6p : 6p^2 : 1 + p^3$. A better parameterization is :*

$$M \simeq 3\left(1+q\right)\left(1-q\right)^2 : 3\left(1+q\right)^2\left(1-q\right) : 3\, q^2 + 1$$

Then the tangent Δ_M at $M \in \mathcal{F}$ is given by

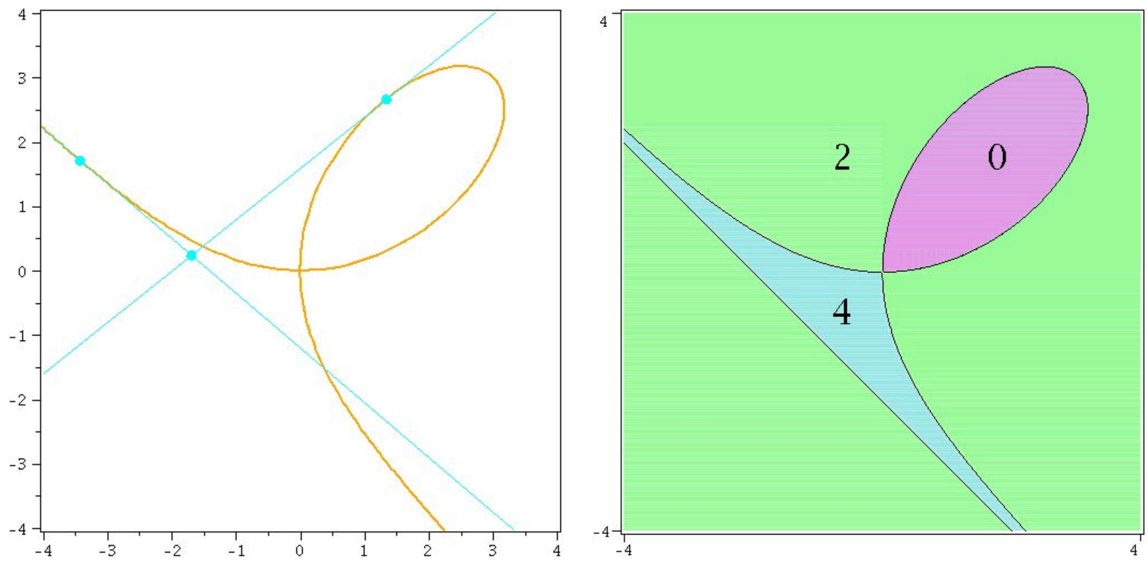
$$N = x : y : z \in \Delta_M \quad when \quad \left[3X^2 - 6T,\, 3Y^2 - 6T,\, -6XY\right] \cdot \left(\begin{array}{c} x \\ y \\ z \end{array}\right) = 0$$

Proof. Homography $q = (p-1)\,/\,(p+1)$ has been used to move point $p = -1$ at $q = \infty$ in order to have a "one piece" curve. Tangency condition is $\overrightarrow{grad\phi} \cdot \overrightarrow{MN} = 0$. But, due to the Darboux property, we already have $\overrightarrow{grad\phi} \cdot M = 0$. □

Example 9.2.3. When $q_1 = 1/3$ and $q_2 = -3$, we obtain points $M_1 = 4 : -5 : 8$ and $M_2 = 24 : -12 : 7$. Tangents are $\left[\begin{array}{ccc} 4 & -5 & 8 \end{array}\right]$, $\left[\begin{array}{ccc} 17 & 20 & 24 \end{array}\right]$ while their common point is (see Figure 9.1) given by :

$$\left[\begin{array}{ccc} 4 & -5 & 8 \end{array}\right] \wedge \left[\begin{array}{ccc} 17 & 20 & 24 \end{array}\right] = 56 : -8 : -33 \approx -1.70 : +0.24 : 1$$

Visible asymptote is the tangent at the visible $T = 0$ point, i.e. at $+1 : -1 : 0$. Using the gradient at that point, we see that asymptote is $[3, 3, -6] \simeq [1, 1, -2]$.



(a) Two intersecting tangents (b) Dividing the plane in three regions

Figure 9.1: Folium of Descartes

Proposition 9.2.4. *Tangential equation of the folium (i.e. the condition for a line $\Delta \simeq [u, v, w]$ to be tangent to the curve is :*

$$\mathcal{F}^*(u, v, w) \doteq 48\,u^2v^2 - 32\,w\left(u^3 + v^3\right) + 24\,uvw^2 - w^4 = 0$$

Proof. Tangency requires a contact point, so that :

$$\left\{(q-1)\left(3\,q^3 + 9\,q + 3\,q^2 + 1\right) = Kv,\; (q+1)\left(3\,q^3 + 9\,q - 3\,q^2 - 1\right) = Ku,\; 6\,(q+1)^2\,(q-1)^2 = Kw\right\}$$

is required. Apart from $w = 0$ or $q = \pm 1$, last equation gives a K value, that can be substitued into the other equations. Writing that the remaining two polynomials have the same roots, we obtain the resolvant :

$$\begin{vmatrix} -w-6\,v & -9\,w-6\,v & -3\,w+6\,v & -3\,w+6\,v & 0 & 0 \\ 0 & -w-6\,v & -9\,w-6\,v & -3\,w+6\,v & -3\,w+6\,v & 0 \\ 0 & 0 & -w-6\,v & -9\,w-6\,v & -3\,w+6\,v & -3\,w+6\,v \\ w+6\,u & -9\,w-6\,u & 3\,w-6\,u & -3\,w+6\,u & 0 & 0 \\ 0 & w+6\,u & -9\,w-6\,u & 3\,w-6\,u & -3\,w+6\,u & 0 \\ 0 & 0 & w+6\,u & -9\,w-6\,u & 3\,w-6\,u & -3\,w+6\,u \end{vmatrix}$$

Suppressing the non vanishing factors leads to the given result. □

Example 9.2.5. Start from a point $N \simeq x:y:z$ and search the u,v,w such that $\{ux+vy+wz=0,\ \Psi=0\}$. We have three different possibilities, that are exemplified by :

$$N \simeq 1:1:1, \quad \begin{array}{ccc} u & v & w \\ 1.0 & -0.51903-1.16372\,i & -0.48097+1.16372\,i \\ 1.0 & -0.31967+0.71674\,i & -0.68033-0.71674\,i \\ 1.0 & -0.51903+1.16372\,i & -0.48097-1.16372\,i \\ 1.0 & -0.31967-0.71674\,i & -0.68033+0.71674\,i \end{array}$$

$$N \simeq 4:2:1, \quad \begin{array}{ccc} u & v & w \\ 1.0 & 1.0 & -6.0 \\ 1.0 & -1.52334 & -0.95333 \\ 1.0 & -0.73833-1.09791\,i & -2.52334+2.19582\,i \\ 1.0 & -0.73833+1.09791\,i & -2.52334-2.19582\,i \end{array}$$

$$N = -\frac{1}{2}:-\frac{1}{2}:1, \quad \begin{array}{ccc} u & v & w \\ 1.0 & 1.35307 & 1.17653 \\ 1.0 & 0.73906 & 0.86953 \\ 1.0 & -0.43584 & 0.28208 \\ 1.0 & -2.29442 & -0.64721 \end{array}$$

In other words, the curve and its asymptote are dividing the plane into three zones. From a magenta point (see Figure 9.1b) no tangents can be drawn to the curve, but two from a green point and four from a cyan point.

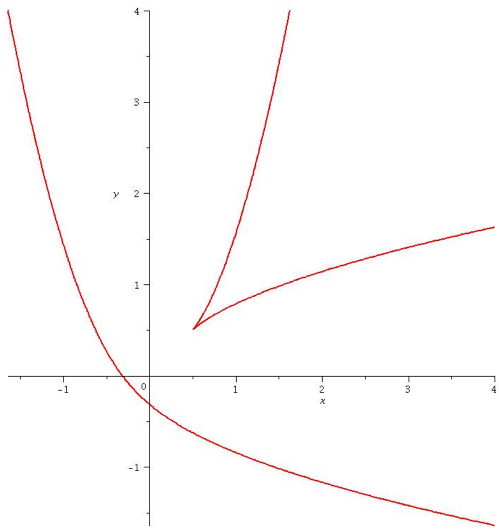


Figure 9.2: Dual curve of the folium

Proposition 9.2.6 (Plucker formulas)**.** *Let d, δ, κ be respectively the degree of a curve $\mathcal{C}$, its number of nodes (double points with two tangents), its number of cups (double point with a single tangent) and d', δ', κ' be the corresponding numbers for the dual curve $\mathcal{C}'$ then we have the following relations :*

$$\begin{aligned} d' &= d\,(d-1) - 2\delta - 3\kappa \\ \kappa' &= 3d\,(d-2) - 6\delta - 8\kappa \\ d &= d'\,(d'-1) - 2\delta' - 3\kappa' \\ \kappa &= 3d'\,(d'-2) - 6\delta' - 8\kappa' \\ g &\doteq \frac{1}{2}\,(d-1)\,(d-2) - \delta - \kappa = \frac{1}{2}\,(d'-1)\,(d'-2) - \delta' - \kappa' \end{aligned}$$

Proof. Proof is not obvious since the formula turns weird when points with multiplicity greater than two are occuring. There is no simpler formula giving δ than $g = g'$. A remark : we have $3d - \kappa = 3d' - \kappa'$. Application : see Figure 9.2. Cups are occuring at $x = y = 1/2$ and at $x = j/2$, $y = j^2/2$ and conjugate. □

9.3 General facts about conics

Definition 9.3.1. A conic $\mathcal{C}$ is a curve whose barycentric equation is an homogeneous polynomial of second degree. This can be written using the usual matrix apparatus :

$$X \in \mathcal{C} \Longleftrightarrow {}^tX\,.\boxed{\mathcal{C}}\,.\,X = (x,y,z)\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Proposition 9.3.2. *Let* $\boxed{M}$ *be a* $n \times n$ *matrix and* $\boxed{M^*}$ *the matrix of the cofactors (at the right place, such that cofactors of a row form a column). Then :*

$$\boxed{M}\,.\,\boxed{M^*} = \boxed{M^*}\,.\,\boxed{M} = \det\left(\boxed{M}\right)\,\mathbf{1}_n$$

Moreover rank $\boxed{M^*} = n$ *when* rank $\boxed{M} = n$, rank $\boxed{M^*} = 1$ *when* rank $\boxed{M} = n-1$, *and* otherwise $\boxed{M^*}$ is the 0 matrix.

Proof. When $\det \boxed{M} \neq 0$, both matrices can be inverted. In the second case a row of $\boxed{M^*}$ describes the wedge of the hyperplane of the columns of $\boxed{M}$. In the last one, all minors are 0 since rank is less than $n-1$. □

Definition 9.3.3. A quadratic form can be written as the sum of as many squares of independent linear forms as its rank, leading to the following classification :

1. When rank is one, $\mathcal{C}$ is a straight line, whose points are each counted twice (strange object).
2. When rank is 2, $\mathcal{C}$ is the union of two intersecting lines. When these lines are complex conjugate of each other (not real), only their intersection is real and this point appears as an isolated point. When one of these lines is the infinity line, $\mathcal{C}$ is considered as some kind of extended circle (see Chapter 10).
3. When $\det \boxed{\mathcal{C}} \neq 0$, intersection of $\mathcal{C}$ and any straight line contains exactly two points (real or not, may be a double point). A such conic is called a proper conic.

Definition 9.3.4. Pole and polar. According to the general definitions, the polar of point X wrt proper conic $\mathcal{C}$ is the line ${}^tX\,\boxed{\mathcal{C}}$, i.e. the locus of the points Y such that ${}^tX\,\boxed{\mathcal{C}}\,Y = 0$, while the pole of line $\{Y \mid \Delta\, Y = 0\}$ is the point $X = \operatorname{adj}\left(\boxed{\mathcal{C}}\right){}^t\Delta$.

Remark 9.3.5. This polarity is not to be confused with polarity wrt the main triangle. Therefore, it can be useful to describe line ${}^tX\,\boxed{\mathcal{C}}$ as the conipolar of X and line ${}^t\mathit{isotom}\,(X)$ as the tripolar of X.

Remark 9.3.6. The relation "point U belongs to the polar of P" is symmetric. When point P belongs to $\mathcal{C}$, its polar wrt the conic is nothing but the line tangent at X to the conic.

Definition 9.3.7. The **dual of a conic** $\mathcal{C}_1$ is the conic $\mathcal{C}_2$ such that point $x:y:z$ belongs to $\mathcal{C}_2$ when line (x,y,z) is tangent to $\mathcal{C}_1$. When dealing with proper conics, we have $\boxed{\mathcal{C}_2} = \operatorname{adj}\left(\boxed{\mathcal{C}_1}\right)$ and conversely. When rank is 2, the dual is rank 1 : all tangents have to pass through the common point.

Definition 9.3.8. The **center** U of a conic $\mathcal{C}$ is the pole of the line at infinity $\mathcal{L}_\infty$ with respect to the conic. Its barycentrics are :

$$\begin{array}{l} -m_{23}^2 + (m_{13} + m_{12})\, m_{23} + m_{22}m_{33} - m_{13}m_{22} - m_{12}m_{33} \\ -m_{13}^2 + (m_{12} + m_{23})\, m_{13} + m_{33}m_{11} - m_{12}m_{33} - m_{23}m_{11} \\ -m_{12}^2 + (m_{23} + m_{13})\, m_{12} + m_{11}m_{22} - m_{23}m_{11} - m_{13}m_{22} \end{array}$$

Definition 9.3.9. A **parabola** is a conic whose center is at infinity (more about parabolae in Section 9.12). Two parallel lines make a non proper parabola. The union of the line at infinity and another line is ... some kind of circle rather than a "special special" parabola.

Fact 9.3.10. *When a conic goes through its center and this center is not at infinity, the conic is the union of two different straight lines. When a conic is a single line whose points are counted twice,* $\det \mathcal{C}$ *vanishes and center has no meaning.*

Proposition 9.3.11. *When* $\mathcal{C}$ *is not a parabola, its center is the symmetry center of the conic.*

Computed Proof. Substitute (3.1) into the equation and obtain $\mathcal{C}(x, y, z)$ times the square of the condition to be a parabola. □

Proposition 9.3.12. *The polar line of a point* P *wrt a conic* $\mathcal{C}$ *is the locus of the points* $Y'Z \cap YZ'$ *where* YY' *and* ZZ' *are chords of* $\mathcal{C}$ *that meet in* X*.*

Proof. Parametrize the line at infinity. Cut the conic by line from P to $U(\mu)$, obtain two points depending of a radical $W(\mu)$. Repeat for another value τ of the parameter. Then check that determinant of lines $Y'Z$, ZY' and ${}^tP\boxed{\mathcal{C}}$ is zero. □

Lemma 9.3.13. *Let* A, B *be some columns and* M *an inversible matrix (dimension* 3 *is assumed). Then*

$$\left({}^tA\,.\,M\right) \wedge \left({}^tB\,.\,M\right) = M^*\,.\,{}^t(A \wedge B)$$

Proof. We have :

$$\begin{aligned} X\,.\,\left({}^tA\,.\,M\right) \wedge \left({}^tB\,.\,M\right) &= \det\left[X\,.\,M^{-1}\,.\,M,\ {}^tA\,.\,M,\ {}^tB\,.\,M\right] \\ &= \det M \times \det\left[X\,.\,M^{-1},\ {}^tA,\ {}^tB\right] \\ &= X\,.\,\left(\det M \times M^{-1}\right)\,.\,{}^t(A \wedge B) \end{aligned}$$

□

Proposition 9.3.14. ***Perspector of a conic****. Let triangle* $A'B'C'$ *be described by the columns of matrix* $\boxed{\mathcal{T}}$*. Then trigone* $A'B'C'$ *is described by the lines of* $\boxed{\mathcal{T}^*}$ *and the triangle of the conipolars of the sidelines are described by the columns of* $\boxed{\mathcal{C}^*}\,.\,{}^t\boxed{\mathcal{T}^*}$ *(remember,* $a*$ *denotes the adjoint matrix). Consider specially the case of triangle* ABC *itself. Either triangle* ABC *is "autopolar" or both triangles are perspective, with perspector* P *given by :*

$$P = \frac{1}{m_{11}m_{23} - m_{13}m_{12}} : \frac{1}{m_{22}m_{13} - m_{23}m_{12}} : \frac{1}{m_{12}m_{33} - m_{13}m_{23}}$$

Proof. First part is from previous lemma, and the simplicity of the result from the special choice of the triangle. □

Proposition 9.3.15. *Let* P *be a point not on the sidelines of* ABC*. Six points are obtained by intersecting a sideline with a parallel through* P *to another sideline. These points are on the same conic* $\mathcal{C}$*. Equation, perspector* T *and center* U *are :*

$$\begin{aligned} \mathcal{C} &= \sum(q+r)qrx^2 - \sum(p^2+pq+pr+2qr)pyz \\ T &= \frac{p}{2\,pr+2\,pq+qr} : \frac{q}{2\,pq+pr+2\,qr} : \frac{r}{pq+2\,pr+2\,qr} \\ U &= p\left(2\,qr+pr+pq-p^2\right) : q\left(2\,pr+pq+qr-q^2\right) : r\left(2\,pq+pr+qr-r^2\right) \end{aligned}$$

Center U *is at infinity (and* $\mathcal{C}$ *is a parabola) when* P *is at infinity or on the Steiner inconic. Points* P *and* $Q = p\,(q+r-p) : q\,(r+p-q)\,, r\,(p+q-r)$ *are leading to the same center* U*. Point* Q *is at infinity when* P *is on the Steiner inconic.*

Proof. Equation in Q is of third degree. The discriminant factors into minus a product of squares. Other computations are straightforward. Examples are [P,U], [115, 523], [1015, 513], [1084, 512], [1086, 514], [1146, 522], [2482, 524], [3163, 30] where P is on the Steiner inconic and [P,U], [2, 2], [6, 182], [3, 182], [9, 1001], [1, 1001], [190, 1016], [664, 1275] for other points. □

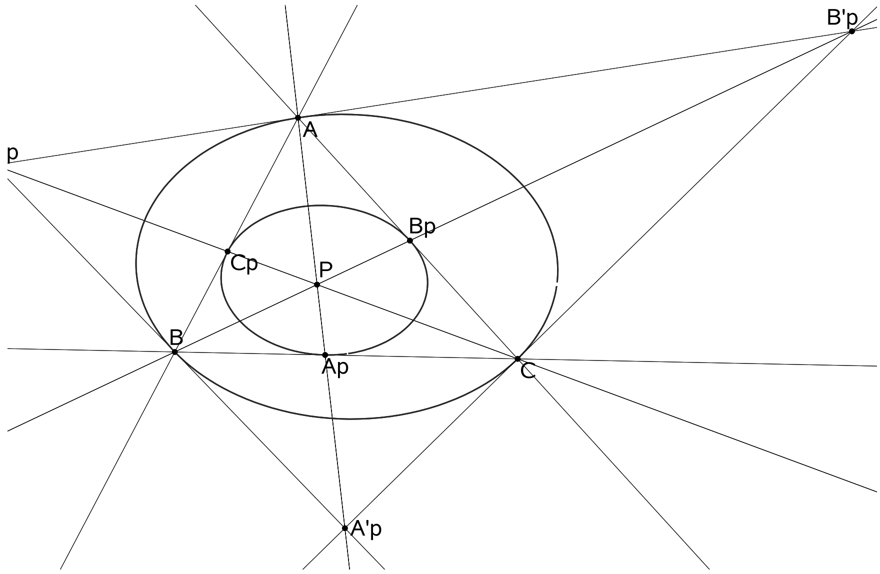


Figure 9.3: Inconic, circumconic and their perspector

9.4 Circumconics

Definition 9.4.1. A circumconic is a conic that contains the vertices A, B, C of the reference triangle. Its equation can be written as :

$$CC(P) = {}^tX \boxed{\mathcal{C}_c} X = 0 \qquad \text{where } \boxed{\mathcal{C}_c} = \begin{pmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{pmatrix} \tag{9.2}$$

Construction 9.4.2. *Graphical tools can construct any conic from five points. Given the perspector P, the other points on the cevian lines are $-p : 2q : 2r$,* etc *i.e points $2qB + 2rC - p\,A$.*

Theorem 9.4.3 (circumconics)**.** *We have the following four properties :*
(i) Point P is the perspector of the conic, and the polar triangle of ABC wrt $CC(P)$ is the anticevian triangle of P wrt ABC (in other words, $CC(P)$ is tangent at A to $P_B P_C$ etc.
(ii) When U is the center of $CC(P)$ then P is the center of $CC(U)$. Both are related by :

$$U = cevadiv(X_2, P) = P *_b anticomplem(P)$$

(iii) Circumconic $CC(P)$ is the P isoconjugate of $\mathcal{L}_\infty$. Inter alia*,*

$$X \in circumcircle \Longleftrightarrow isog(X) \in \mathcal{L}_\infty$$

(iv) The polar line of U wrt $CC(P)$ is $\simeq [qw + rv,\, ru + pw,\, pv + qu]$ ... aka the crosssumbar of (P, U).

Proposition 9.4.4. ***Points at infinity****. A circumscribed conic is an ellipse, a parabola or an hyperbola when its perspector is inside, on or outside the Steiner in-ellipse. Moreover, its points at infinity, expressed from the perspector $Q = p : q : r$ have the following barycentrics :*

$$M_\infty \simeq \begin{pmatrix} -2\,p \\ p + q - r - W \\ p + r - q + W \end{pmatrix} \qquad \text{where } W^2 = p^2 + q^2 + r^2 - 2\,pq - 2\,qr - 2\,rp$$

When the point at infinity of a circumparabola is $u : v : w$, its perspector is $P = u^2 : v^2 : w^2$.

Proof. Immediate computation. Mind the fact that $W^2 = -3$ when P is at X(2). For the second part, start from $T = u : v : -u - v$, and compute the circumconic relative to $u^2 : v^2 : (u + v)^2$. $\square$

Proposition 9.4.5. *When a circumscribed conic is a rectangular hyperbola, its perspector is on the tripolar of H =X(4), while its center is on the Euler circle.*

Proof. Write that trace $\left(\boxed{\mathcal{M}}\boxed{\mathcal{C}}\right) = 0$ and obtain a first degree equation for the perspector, then substitute the perspector from the center. (2) We have $U \in \mathcal{L}_\infty$ and the $U = cevadiv(X_2, P)$ formula. □

Proposition 9.4.6. *Two (non equal) circumconics $CC(P)$ and $CC(U)$ have exactly four points in common, namely the three vertices and the tripolar of line PU. Formally, this point is $isotom(P \wedge U)$.*

Proof. Conics that share five distinct points are equal. The value of X follows by direct inspection. □

Proposition 9.4.7. *The perspector P of the circumconic through additional points U_1, U_2 is the intersection of the tripolars of U_1 and U_1 (caveat: this is not the tripole of line U_1U_2). In other words, :*

$$P = U_1 *_b U_2 *_b (U_1 \wedge U_2)$$

Proof. Direct inspection. □

Remark 9.4.8. Collineations can be used to transform any circumconic into the circumcircle or the Steiner outellipse, so that many proofs can be done assuming such special cases. More details in Proposition 13.4.3.

9.5 Inconics

Definition 9.5.1. An inconic is a conic that is tangent to all three sides of the reference triangle.

Theorem 9.5.2 (inconics). *The punctual and tangential equations of an inconic $\mathcal{C}_i$ are :*

$$^tX\,.\boxed{\mathcal{C}_i}.\,X,\ \Delta\,.\boxed{\mathcal{C}_i^*}.\,{}^t\Delta \qquad \text{where } \boxed{\mathcal{C}_i} \simeq \begin{bmatrix} p^2 & -pq & -pr \\ -pq & q^2 & -rq \\ -pr & -rq & r^2 \end{bmatrix},\ \boxed{\mathcal{C}_i^*} \simeq \begin{bmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{bmatrix} \tag{9.3}$$

Point $Q = p : q : r$ is called the auxiliary point of the conic. The dual conic of $\mathcal{C}_i$ is precisely $CC(Q)$. The contact points are $0 : r : q$, etc. They are the cevians of point $P = isotom(Q)$. This point is the perspector between triangle ABC and its polar triangle with respect to the conic. The center of $\mathcal{C}_i$ is the complement of the auxiliary point.

Proof. By definition, $\boxed{\mathcal{C}_i^*}$ must have a zero diagonal. Then $\boxed{\mathcal{C}_i}$ is its adjoint. Perspectivity is obvious, while center is the pole of the line at infinity. □

Remark 9.5.3. Dual of $CC(Q)$ is $IC(Q)$ means that Q is the perspector of the CC and the auxiliary point of the IC.

Corollary 9.5.4. *Direct relations between center and perspector are as follows :*

$$\begin{array}{rclclcl} U &=& complem\,(isot\,(P)) &=& crossmul(X_2, P) &=& P *_b complem\,(P) \\ P &=& isot\,(anticomplem\,(U)) &=& crossdiv\,(U, X_2) && \end{array}$$

Construction 9.5.5. *An inscribed conic can be generated as follows. Given the perspector P, draw the cevians, obtain $A_PB_PC_P$ and draw the cevian triangle. Draw an arbitrary line Δ through B. Define $B_a = (B_PA_P) \cap \Delta$ and $B_c = (B_PC_P) \cap \Delta$. Then $M = B_aC_P \cap B_cA_P$ is on the conic. Thereafter, you can either :*
(i) define Δ by a point X on the circumcircle, and generate $\mathcal{C}_i$ as the locus of M (parameterization by a turn)
(ii) repeat the construction from C, obtaining N and draw $\mathcal{C}_i$ from the five points $A_PB_PC_P\,M\,N$.

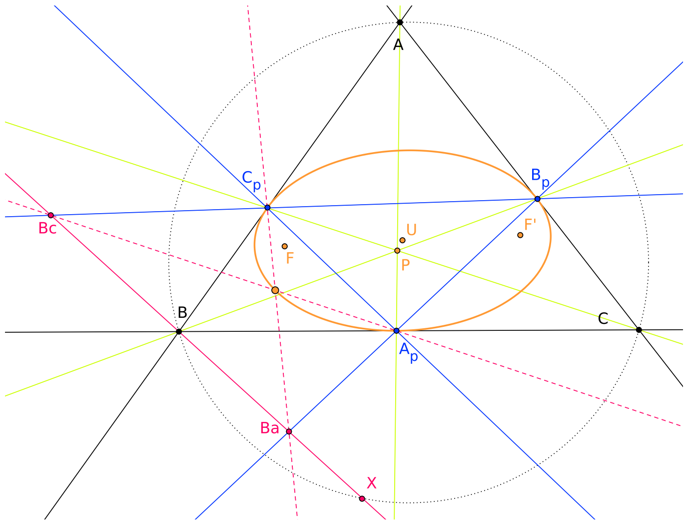


Figure 9.4: How to generate an inscribed conic from its perspector.

Proof. Use $X = x : y : z$ so that $\Delta \simeq [-z, 0, x]$. Thereafter :

$$B_a \simeq \left(\begin{array}{c} cqx \\ -px + rz \\ qz \end{array} \right), \; B_c \simeq \left(\begin{array}{c} qx \\ px - rz \\ qz \end{array} \right), \; M \simeq \left(\begin{array}{c} pq\, x^2 \\ (px - rz)^2 \\ qr\, z^2 \end{array} \right) \qquad \square$$

Proposition 9.5.6. ***Points at infinity.*** *An inscribed conic is an ellipse, a parabola or an hyperbola when its perspector is inside, on or outside the Steiner circum-ellipse. Moreover, its points at infinity, expressed from the auxiliary point $Q = p : q : r$ have the following barycentrics :*

$$M_\infty \simeq \left(\begin{array}{c} (r+q)^2 \\ pq - r\,(p+q+r) + 2\,W \\ pr - q\,(p+q+r) - 2\,W \end{array} \right) \qquad \text{where } W^2 = -pqr\,(p+q+r)$$

Proof. Immediate computation. Steiner property comes from $W^2 = -p^2q^2r^2 \left(\frac{1}{q}\frac{1}{r} + \frac{1}{r}\frac{1}{p} + \frac{1}{p}\frac{1}{q} \right)$. $\square$

Proposition 9.5.7. *An inscribed conic is a rectangular hyperbola if, and only if, its auxiliary point is on the Longchamps circle. Equivalently, its center is on the conjugate circle. See Section 10.7 and Section 10.8.*

Proof. Write that trace $\left(\boxed{\mathcal{M}} \boxed{\mathcal{C}} \right) = 0$, see Proposition 9.13.4. $\square$

Proposition 9.5.8. *When center U is given, the perspectors P_i, P_c of the corresponding in- and circum-conics are related by :*

$$P_i *_b P_c = U \qquad ; \qquad P_i = anticomplem\,(P_c)$$

When perspector P is given, the centers U_i, U_c are aligned with $P = p : q : r$ together with $p^2 : q^2 : r^2$.

Proof. Direct inspection $\square$

Proposition 9.5.9. *Let P be a fixed point, not on the sidelines, and U be a point moving point on tripolar (P). The envelope of all the lines $\Delta = tripolar\,(U)$ is the inconic $IC\,(P)$. Moreover, the contact point of Δ is $T = U *_b U \div_b P$.*

Proof. Yet given result. But now, this is the right place to prove it. Write :

$$\begin{array}{rcl} P &=& p:q:r \\ U &=& p\,(\sigma-\tau):q\,(\tau-\rho):r\,(\rho-\sigma) \\ \Delta &\simeq& (1\div p(\sigma-\tau):1\div q(\tau-\rho):1\div r(\rho-\sigma)) \\ T &=& p\,(\sigma-\tau)^2:q\,(\tau-\rho)^2:r\,(\rho-\sigma)^2 \end{array}$$

and check that : $\Delta\,.\,\mathrm{adj}\left(\boxed{\mathcal{C}_i}\right).{}^t\Delta=0$, $\Delta\,.\,T=0$ and ${}^tT\,.\boxed{\mathcal{C}_i}\,.T=0$ where $\boxed{\mathcal{C}_i}$ is given in (9.3) ☐

Proposition 9.5.10. *Let $\Delta\simeq[\rho,\sigma,\tau]$ be a line that cuts the sidelines BC, CA, AB in A',B',C' and the circumconic $K=CC\,(P)$ where $P=p:q:r$. Let $X=x:y:z$ be a point on Δ, and A" the other intersection of XA with K and cyclically B" $\in XB$, C" $\in XC$. Then lines $A'A$", $B'B$", $C'C$" are concurrent on a point $Q\in K$. Moreover,*

$$Q=\frac{p}{\rho x}:\frac{q}{\sigma y}:\frac{r}{\tau z}$$

Proof. Use parameterization $X=\Delta\wedge U$ where U is a moving line. ☐

9.6 Some in and circum conics

A list of specific in- and circum- conics is given in Table 9.1. Kiepert RH is studied at Brocard Section.

P	IC	center	CC	center
X_1		X_{37}		X_9
X_2	inSteiner 9.10	X_2	circumSteiner	X_2
X_3		X_{216}	outMacBeath 9.6.3	X_6
X_4	orthic	X_6		X_{1249}
X_5		X_{233}		X_{216}
X_6	Brocard 9.11	X_{39}	circumcircle 10.4	X_3
X_7	incircle 10.5	X_1		X_{3160}
X_8	Mandart	X_9		X_{3161}
X_{99}	Kiepert parabola 12.4.3	IX_{523}		???
X_{190}	Yff parabola	IX_{514}		???
X_{264}	inMacBeath 9.6.2	X_5		???
∞X_{523}		X_{115}	Kiepert RH 10.18.1	X_{115}
X_{598}	inLemoine	X_{597}		???
X_{647}		???	Jerabek RH	X_{125}
X_{650}		???	Feuerbach RH	X_{11}

RH is rectangular hyperbola
The Feuerbach RH is the isogonal conjugate of the line (circumcenter,incenter).
The Jerabek RH is the isogonal conjugate of the Euler line.

Table 9.1: Some Inconics and Circumconics

Example 9.6.1. The **Steiner** ellipses (centers=perspector=X_2) are what happen to both the circum- and in-circle of an equilateral triangle when this triangle is transformed into an ordinary triangle by an affinity. The Steiner circumellipse (S) is the isotomic conjugate of $\mathcal{L}_\infty$ and the isogonal conjugate of the Lemoine axis. Since $isog\,(\mathcal{L}_\infty)$ is the circumcircle (C), the mapping $X\mapsto X*_b X_6$ sends (S) onto (C). Steiner in-ellipse is the envelope of the line whose tripole is at infinity (more about Steiner inellipse in Section 9.10).

Example 9.6.2. The **MacBeath-inconic** was introduced as follows:

> Lemma 1: Let O, H be the common points of a coaxal system of circles. Let a variable circle of the system cut the line of centers at C. Let T be a point on the circumference such that $TC = k * OC$, where k is a fixed ratio. Then the locus of T is a conic with foci at O, H (Macbeath, 1949)

Its perspector is X(264), center X(5) and foci X(3) and X(4) . Its barycentric equation is :

$$\sum \frac{a^4\left(-a^2+b^2+c^2\right)x^2}{\left(c^2+a^2-b^2\right)\left(a^2+b^2-c^2\right)} - 2\sum \frac{b^2c^2yz}{b^2+c^2-a^2}$$

and this conic goes through X(I) for I =

$$339, 1312, 1313, 2967, 2968, 2969, 2970, 2971, 2972, 2973, 2974$$

Example 9.6.3. The **MacBeath circumconic**, is the dual to the MacBeath inconic. Its perspector is X(3) and its center X(6). Its barycentric equation is :

$$a^2(b^2+c^2-a^2)yz + b^2(c^2+a^2-b^2)zx + c^2(a^2+b^2-c^2)xy = 0$$

and it goes through X(I) for I =

$$110, 287, 648, 651, 677, 895, 1331, 1332, 1797, 1813, 1814, 1815$$

9.7 Cevian conics

Proposition 9.7.1. *For any points $P = p : q : r$ and $U = u : v : w$, not on a sideline of ABC, the cevians of P and U are on a same conic, whose equation in $x : y : z$ is $conicev\,(P, U)$ given by :*

$$\frac{1}{up}x^2 + \frac{1}{qv}y^2 + \frac{1}{rw}z^2 - \left(\frac{1}{qu}+\frac{1}{vp}\right)yx - \left(\frac{1}{ru}+\frac{1}{wp}\right)zx - \left(\frac{1}{rv}+\frac{1}{qw}\right)zy = 0$$

Proof. Apply $(x, y, z) \to \left[x^2, xy, y^2, yz, z^2, zx\right]$ to the six points and check that rank isn't 6. □

Example 9.7.2. Any inconic is a cevian conic : $IC\,(P) = conicev\,(P, P)$. For example, the incircle is $IC\,(X_7) = conicev\,(X_7, X_7)$.
A non trivial example is the nine points circle, aka $conicev\,(X_2,\ X_4)$.

Proposition 9.7.3. *Assume that a conic $\mathcal{C}$ encounters the sidelines of ABC in six (real) points, none of them being a vertex A, B, C. Three of these points are the cevians of some point P if and only if :*

$$\begin{aligned} &m_{33}m_{22}m_{11} - m_{11}\,m_{23}^2 - m_{22}\,m_{13}^2 - m_{33}\,m_{12}^2 - 2\,m_{13}m_{23}m_{12} \\ &\qquad = \det M - 4\,m_{13}m_{23}m_{12} = 0 \end{aligned}$$

In this case, the remaining three intersections are the cevians of some point U and both P, U_are given by :

$$P, U \simeq \left[\begin{array}{c} \left(+m_{13}m_{12} + m_{11}m_{23} + m_{13}\sqrt{m_{12}^2 - m_{22}m_{11}}\right)/m_{11} \\ \left(-m_{22}m_{13} - m_{23}m_{12} + m_{23}\sqrt{m_{12}^2 - m_{22}m_{11}}\right)/m_{22} \\ -\sqrt{m_{12}^2 - m_{22}m_{11}} \end{array}\right]$$

Proof. Hypotheses are implying $m_{11} \neq 0$ ($A \notin \mathcal{C}$) and $m_{12}^2 - m_{22}m_{11} \geq 0$ (existence of intersections). □

Proposition 9.7.4. *The fourth common point F between cevian conics $conicev\,(P, U_1)$ and $conicev\,(P, U_2)$ can be obtained from the tripolars of U_1, U_2. We have :*

$$\begin{aligned} X &\simeq tripolar\,(U_1) \cap tripolar\,(U_2) = (U_1 \wedge U_2) *_b U_1 *_b U_2 \\ F &\simeq anticomplem\,(X \div_b P) *_b X \end{aligned}$$

Proof. Direct inspection. □

9.8 Direction of axes

Proposition 9.8.1. *Let $\mathcal{C}$ be a conic, but not a circle, and γ an auxiliary circle. Consider, in any order, the common points X_1, X_2, X_3, X_4 of $\mathcal{C}$ and γ. Then axes of $\mathcal{C}$ have the same directions as bisectors of angle $\left(\overbrace{X_1X_2,\ X_3X_4}\right)$.*

Proof. Use rectangular Cartesian coordinates. Then $\mathcal{C}$ is $y^2 = 2px + qx^2$ and γ is $(x-a)^2 + (y-b)^2 = r^2$. Substitution $y^2 = Y$ gives :

$$2\,by = (1+q)\,x^2 + (2\,p - 2\,a)\,x + b^2 + a^2 - r^2 \quad ; \quad Y = 2px + qx^2$$

By substitution and reorganization :

$$\frac{y_2 - y_1}{x_2 - x_1} + \frac{y_4 - y_3}{x_4 - x_3} = \frac{2\,p - 2\,a}{b} + \frac{(1+q)}{2b}\,(x_1 + x_2 + x_3 + x_4)$$

Value of $\sum x_i$ is obtained from fourth degree equation $0 = Y - y^2 = (1+q)^2\,x^4 - 4\,(1+q)\,(a-p)\,x^3 \cdots$. This proves that X_1X_2 and X_3X_4 are symmetrical wrt the axes and conclusion follows. By the way, it has been proven that points X_i can be sorted in any order without changing the result. □

Proposition 9.8.2. *When $\mathcal{C}$ is a circumconic, but not the circumcircle itself, directions of axes are given by the bisectors of $\left(\overbrace{BC,\ AM}\right)$ where M is the fourth common point of $\mathcal{C}$ and Γ. When $\mathcal{C}$ is not a circumconic, it exists nevertheless an unique point $M \in \Gamma$ so that axes of $\mathcal{C}$ have the same directions as the bisectors of $\left(\overbrace{BC,\ AM}\right)$. We have formulae :*

$$\mathcal{C} = \begin{pmatrix} l & h & g \\ h & m & f \\ g & f & n \end{pmatrix} \mapsto \begin{pmatrix} 1/\left((l+m-2\,h)\,b^2 - (n+l-2\,g)\,c^2\right) \\ 1/\left((m+n-2\,f)\,c^2 - (l+m-2\,h)\,a^2\right) \\ 1/\left((n+l-2\,g)\,a^2 - (m+n-2\,f)\,b^2\right) \end{pmatrix}$$

$$CC\,(P) \mapsto \begin{pmatrix} 1/\left(rb^2 - qc^2\right) \\ 1/\left(pc^2 - ra^2\right) \\ 1/\left(qa^2 - pb^2\right) \end{pmatrix} \quad ; \quad IC\,(U) \mapsto \begin{pmatrix} 1/\left(w^2b^2 - v^2c^2\right) \\ 1/\left(u^2c^2 - w^2a^2\right) \\ 1/\left(v^2a^2 - u^2b^2\right) \end{pmatrix}$$

Proof. First part is preceeding proposition. For the second part, we have :

$$\begin{aligned} \tan\left(\overbrace{BC,\ B\delta}\right) - \tan\left(\overbrace{B\delta,\ AM}\right) &= 0 \\ \tan\left(\overbrace{B\Delta_1,\ B\delta}\right) - \tan\left(\overbrace{B\delta,\ B\Delta_2}\right) &= 0 \end{aligned}$$

where Δ_j are the points at infinity of the conic, δ is an unknown direction and M an unknown point. The second equation fixes δ, and this is sufficient to fix the direction of AM, i.e. the y, z barycentrics of M. The x barycentric is obtained from $M \in \Gamma$.

Some technical details. We note $\delta = [u, v, w]$ and $M = p : q : r$. Points Δ_j depend on a square-root :

$$W^2 = (n + h - g - f)^2 - (n + l - 2\,g)\,(m + n - 2\,f)$$

It happens the odd powers of W are cancelling in the second equation. Then we eliminate q and obtain :

$$q = \frac{p\left(w^2b^2 - v^2c^2\right)}{w^2c^2 - w^2a^2 + 2\,wvc^2 + v^2c^2}$$

We extract v^2 as a function of v and the other variables and substitute into the q- expression. This left two first degree factors in v, and they cancel, leading to the result. Another path could be $\left(\overbrace{BC,\ AM}\right) \doteq \left(\overbrace{BC,\ \Delta_1}\right) + \left(\overbrace{BC,\ \Delta_2}\right)$, avoiding the unknown δ, but the Maple-length of the starting expression would be 544846 ! □

Definition 9.8.3. Gudulic point. The preceeding method that specifies a pair of orthogonal directions by giving a point $M \in \Gamma$ such that the bisectors of $\left(\overbrace{BC,\, AM}\right)$ have the required directions was firstly used by Lemoine (1900). The name "gudulic point" was coined in a discussion of `www.les-mathematiques.net`, in honor to St Gudula of Brussell.

9.9 Focuses of an inconic

Definition 9.9.1. A point F is a **focus** for a curve when the isotropic lines from this point are tangent to the curve.

Proposition 9.9.2. *The geometric focuses of a conic are examples of the preceding definition. But a conic has, in the general case, four analytical focuses : the two geometrical ones, and two extra, imaginary, focuses that stay on the other axis.*

Proof. Let us consider ellipse $x^2/a^2 + y^2/b^2 = 1$ and isotropic line $(x - x_0) + i\,(y - y_0) = 0$. Their intersection is given by a second degree equation whose discriminant is $(x_0 + i\,y_0)^2 - \left(a^2 - b^2\right) = 0$. It vanishes when $y_0 = 0$, $x_0 = \pm f$ (as usual) but also when $x_0 = 0$, $y_0 = \pm i\,f$. □

Proposition 9.9.3. *Let $F = p : q : r$ be a point not lying on the sidelines. As in Figure 9.5, we note F_a, F_b, F_c and F'_a, F'_b, F'_c the projections and the reflections of F about the sidelines ; G the isogonal conjugate of F, $\omega = (F + G)/2$ and $P_a = FG'_a \cap F'_aG$, etc. Then points P_a, P_b, P_c are the cevians of $P = (isotom \circ anticomplem)(\omega)$, and are the contact points of $\mathcal{C} \doteq inconic\,(P)$. This conic admits ω as center and F, G as geometrical focuses. Moreover circle $F'_aF'_bF'_c$, centered at G is the circular directrix of this conic wrt focus F while circle $F_aF_bF_c$ is the principal circle (tangent at major axis).*

When either F or G is at infinity, the other is on the circumcircle, P is on the Steiner circumconic, and $\mathcal{C}$ is a parabola.

Proof. One obtains easily :

$$G'_a = -\frac{a^4}{p} : \frac{S_c a^2}{p} + \frac{a^2b^2}{q} : \frac{S_b a^2}{p} + \frac{a^2c^2}{r}$$

leading to the symmetrical expression :

$$|F'_aG|^2 = \frac{\left(c^2q^2 + 2S_aqr + b^2r^2\right)\left(a^2r^2 + 2S_brp + c^2p^2\right)\left(b^2p^2 + 2S_cpq + a^2q^2\right)}{(r + q + p)^2\left(a^2qr + b^2pr + c^2pq\right)^2}$$

proving that $\gamma \doteq F'_aF'_bF'_c$ is centered at G. When P is inside ABC, we can define an ellipse $\mathcal{C}$ by focus F and circular directrix γ. We have $P_a \in \mathcal{C}$ since $|FP_a| = |F'_aP_a|$. Moreover BC is the bissector of $(P_aF,\, P_aF'_a)$, again by symmetry. Therefore $\mathcal{C}$ is the inscribed conic tangent at P_a, P_b, P_c and we have :

$$\omega \simeq \begin{pmatrix} \left(rb^2 + qc^2\right)p^2 + (2\,p + q + r)\,qra^2 \\ \left(pc^2 + ra^2\right)q^2 + (p + 2\,q + r)\,rpb^2 \\ \left(qa^2 + pb^2\right)r^2 + (p + q + 2\,r)\,pqc^2 \end{pmatrix} \; ; \; P \simeq \begin{pmatrix} \dfrac{rq}{q^2c^2 + 2\,S_arq + r^2b^2} \\ \dfrac{rp}{c^2p^2 + 2\,S_brp + a^2r^2} \\ \dfrac{qp}{b^2p^2 + 2\,S_cqp + q^2a^2} \end{pmatrix}$$

When P is outside ABC, the simplest method is to revert the process and define P, and therefore $\mathcal{C}$, using the given formula and, thereafter, check that lines $F\Omega^{\pm}$ are tangents to $\mathcal{C}$. □

Remark 9.9.4. When substituting $F = p : q : r$ by an umbilic $\Omega^{\pm}$, the ω formula gives $0 : 0 : 0$. Therefore, umbilics are expected to appear as artefacts when trying to revert the ω formula to obtain the foci.

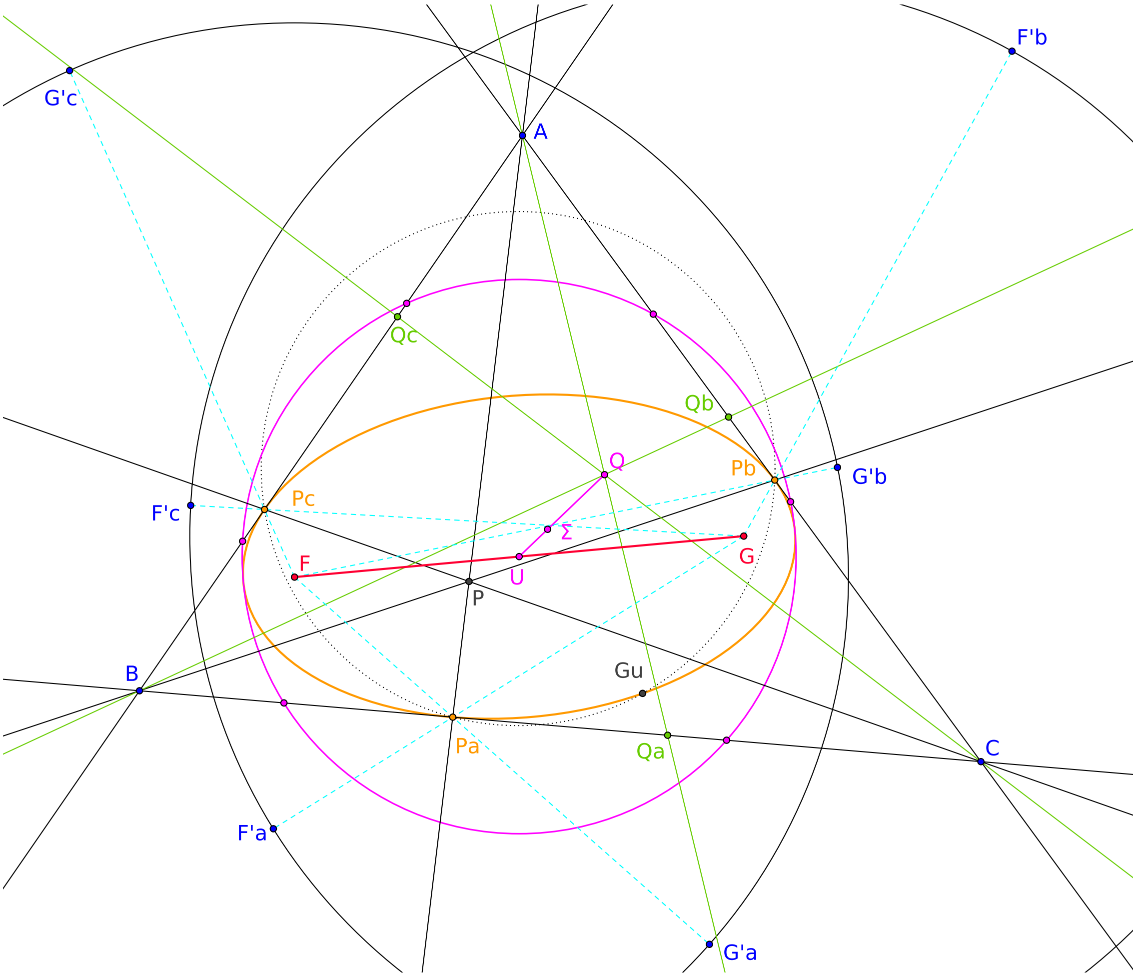


Figure 9.5: Focus of an inconic

Proposition 9.9.5. *The focus of an inconic can be obtained from the perspector by two successive second degree equations (ruler and compass construction). Let $P = p : q : r$ be the perspector. Then the four focuses can be written as :*

$$F_i \simeq \left(\begin{array}{c} p\,(r+q)+\sqrt{K} \\ q\,(r+p)+t\sqrt{K} \\ r\,(q+p)-(1+t)\,\sqrt{K} \end{array} \right)$$

Then t is homographic in K, while K is solution of a second degree equation. The converse is true : K is homographic in t, while t is solution of a second degree equation (with same discriminant). Obviously, the two solutions in t lead to orthogonal directions (orthopoints at infinity)

Proof. Straightforward elimination.

□

Example 9.9.6. Table 9.2 gives some examples of perspectors and focuses. In this table, expressions like $X5 \pm X30$ are not "up to a proportionality factor", but are addressing the usual simplified values. All expressions are "centered" at the center, the $\pm$ term being at infinity. Imaginary focuses associated with perspector X(80) have a very simple expression, namely $a : bj : cj^2$ and $a : bj^2 : cj$ where j is the third of a full turn.

$$\begin{aligned} W_{673} &= \sqrt{a^2+c^2+b^2-2\,bc-2\,ac-2\,ba} \\ W_{694} &= a^2b^2c^2\sqrt{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right)\left(-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \end{aligned}$$

P	ω	F	G	FG	HK	*nom*
6	39	ω^+	ω^-	$X39 \pm X512$	$X39 \pm \dfrac{i}{4S}\, X511$	*Brocard*
7	1	1	1	$X1$	$X1$	*inscrit*
80	44			$X44 \pm \dfrac{\sqrt{3}}{4S}\, X517$	$X44 \pm i\sqrt{3}\, X513$	
264	5	4	3	$X5 \pm X30$	$X5 \pm i4S\, X523$	*MacBeath*
598	597	2	6	$X597 \pm X524$	$X597 \pm \dfrac{i}{4S}\, X1499$	*Lemoine*
673	3008			$X3008 \pm W\, X514$	$X3008 \pm \dfrac{i}{4S}\, W\, X516$	
694	3229			$X3229 \pm W\, X512$	$X3229 \pm \dfrac{i}{4S}\, W\, X511$	

Table 9.2: Perspector and focuses of some inconics

9.10 Focuses of the Steiner inconic

9.10.1 Using barycentrics

Definition 9.10.1. The **Steiner** in-ellipse is what happen to the incircle of an equilateral triangle when this triangle is transformed into an ordinary triangle by an affinity. One has center =perspector $=X_2$. Moreover, the Steiner inconic is the envelope of the lines whose tripoles are at infinity. Equation of this conis is :

$$x^2+y^2+z^2-2xy-2yz-2zx=(x,y,z)\begin{pmatrix} +1 & -1 & -1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$$

Notation 9.10.2. In this section, the following radicals will be used :

$$W=\sqrt{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}$$
$$W_a=\sqrt{\left(W+a^2\right)^2-b^2c^2}\,;\; W_b=\sqrt{\left(W+b^2\right)^2-c^2a^2}\,;\; W_c=\sqrt{\left(W+c^2\right)^2-a^2b^2}$$

Proposition 9.10.3. *All these radicals ares real. Moreover, assuming $c>a$, $c>b$, we have :*

$$W_c=W_a+W_b$$
$$\left(c^2-b^2\right)W_b=W\,W_a+\left(b^2-a^2\right)W_a$$
$$\left(c^2-b^2\right)W_c=W\,W_a+\left(c^2-a^2\right)W_a$$

Proof. Let $\overline{W_a}=\sqrt{\left(-W+a^2\right)^2-b^2c^2}$. Then $\left(W_a\,\overline{W_a}\right)^2=-16S^2\left(b^2-c^2\right)^2<0$, while W_a^2, $\overline{W_a^2}$ are real and $W_a^2>\overline{W_a^2}$. Therefore each W_i is real (and positive). The expression

$$\left(W_a+W_b+W_c\right)\left(-W_a+W_b+W_c\right)\left(W_a-W_b+W_c\right)\left(W_a+W_b-W_c\right)$$

depends only on the W_i^2 and is therefore rational in W. Using the value of W^2, this expression simplifies to 0 : one of the W_i is the sum of the other two. If c is the greatest side, this leads to $W_c=W_a+W_b$. In the same manner, the product :

$$\left(\left(c^2-b^2\right)W_c+\left(c^2-a^2+W\right)W_a\right)\left(\left(c^2-b^2\right)W_c-\left(c^2-a^2+W\right)W_a\right)$$

simplifies to 0. If c is the greatest side, the first factor cannot vanish, leading to the W_c formula. The W_b formula results by substraction. □

Proposition 9.10.4. *The foci of the Steiner inconic are given by $F_{\pm}=X_{3413}\pm Q\,X_2$ where :*

$$X_2=\left(\begin{array}{c}1\\1\\1\end{array}\right)\;;\;X_{3413}=\left(\begin{array}{c}\left(b^2-c^2\right)\left(a^4-b^2c^2-a^2\,W\right)\\\left(c^2-a^2\right)\left(b^4-a^2c^2-b^2\,W\right)\\\left(a^2-b^2\right)\left(c^4-a^2b^2-c^2\,W\right)\end{array}\right)$$
$$Q=\sqrt{2a^2b^2c^2\,W^3-16\,(a^4b^4+b^4c^4+c^4a^4)\,S^2+a^2b^2c^2\,(a^2b^2+b^2c^2+c^2a^2)\,(a^2+b^2+c^2)}$$

Proof. Isotropic lines through a focus are tangent to the curve. Write that $F\Omega^+$ is tangent to $\mathcal{C}$ and separate real and imaginary parts. Eliminate one of the coordinates of F from this system. It remains a fourth degree equation (E) giving the two real and two imaginary foci. The discriminant of this equation contains W^4 in factor. Using this indication, we factorize (E) over $\mathbb{R}(W)$ and obtain :

$$\left(c^2v^2-\left(2\,wa^2+2\,wW\right)v+b^2w^2\right)\left(c^2v^2-\left(2\,wa^2-2\,wW\right)v+b^2w^2\right)=0$$

The discriminants of these second degree factors are W_a^2 and $\left(\overline{W_a}\right)^2$. And we obtain the non symmetric expression :

$$F_+\simeq\left(\begin{array}{c}\left(W+b^2\right)\left(b^2-c^2\right)+\left(W+b^2-a^2\right)W_a\\\left(b^2-c^2\right)\left(W_a+W+a^2\right)\\\left(b^2-c^2\right)c^2\end{array}\right)$$

In order to obtain a more symmetrical expression, one can compute $U=\mathcal{L}_\infty\wedge(F_+\wedge F_-)$, i.e. the point at infinity of the focal line. This point happens to be X_{3413}, the first Kiepert infinity point. The existence of Q is obvious since X_2 is the middle of the foci. A straightforward computation leads to the given formula.

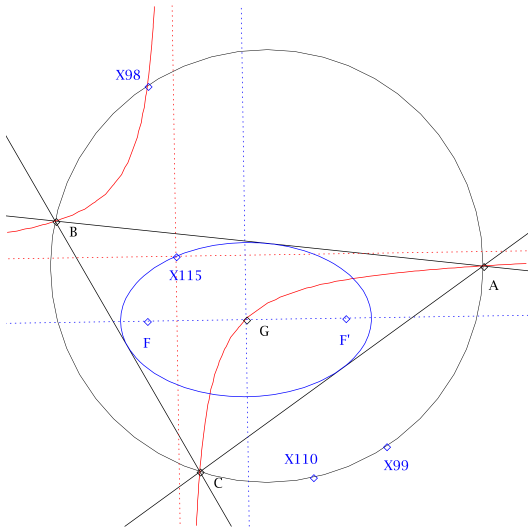


Figure 9.6: The Steiner inellipse

Figure 9.6 summarizes these properties. The hyperbola is Kiepert RH, the Tarry point X(98) is the gudulic point of the KRH axes, while it's circumcircle antipode, the Steiner point X(99), is the gudulic point of both the KRH asymptotes and the Steiner axes. □

9.10.2 Using Morley affixes

Proposition 9.10.5. *Using Morley affixes, the tangential equation of the Steiner inconic is :*

$$\boxed{\mathcal{C}_z^*}\simeq\left(\begin{array}{ccc}2\,\sigma_2\sigma_3&2\,\sigma_1\sigma_3&\sigma_2\sigma_1-3\,\sigma_3\\2\,\sigma_1\sigma_3&6\,\sigma_3&2\,\sigma_2\\\sigma_2\sigma_1-3\,\sigma_3&2\,\sigma_2&2\,\sigma_1\end{array}\right)$$

Proof. Start from barycentric equation and transmute. Ordinary equation is not so handy, and we know that the adjoint matrix will look better. □

Proposition 9.10.6. *The four focuses of the inSteiner ellipse can be written as :*

$$F_j \simeq \left(\begin{array}{c} \sigma_1 \pm W_0\,W_f \\ 3 \\ \dfrac{\sigma_2}{\sigma_3} \pm \dfrac{W_g}{W_0} \end{array}\right) \qquad \text{where } W_0 = \sqrt{\sigma_3},\ W_f = \sqrt{\frac{\sigma_1^2 - 3\,\sigma_2}{\sigma_3}},\ W_f = \sqrt{\frac{\sigma_2^2 - 3\,\sigma_1\sigma_3}{\sigma_3}}$$

The conjugate of W_0 is $1/W_0$, while W_f, W_g are the conjugate of each other. From all the four possibilities for the $\pm$, two of them lead to visible points (the real focuses), the other two lead to non visible points (the analytical focuses).

Proof. Write that isotropic lines $\Omega_\pm F_j$ are tangent to the ellipse. The only difficulty is a sound gestion of the conjugacies. □

9.11 The Brocard ellipse, aka the K-ellipse

Remark 9.11.1. The K-circumconic, i.e. CC(X(6)), is nothing but the circumcircle. Therefore, the K-ellipse is the K-inconic.

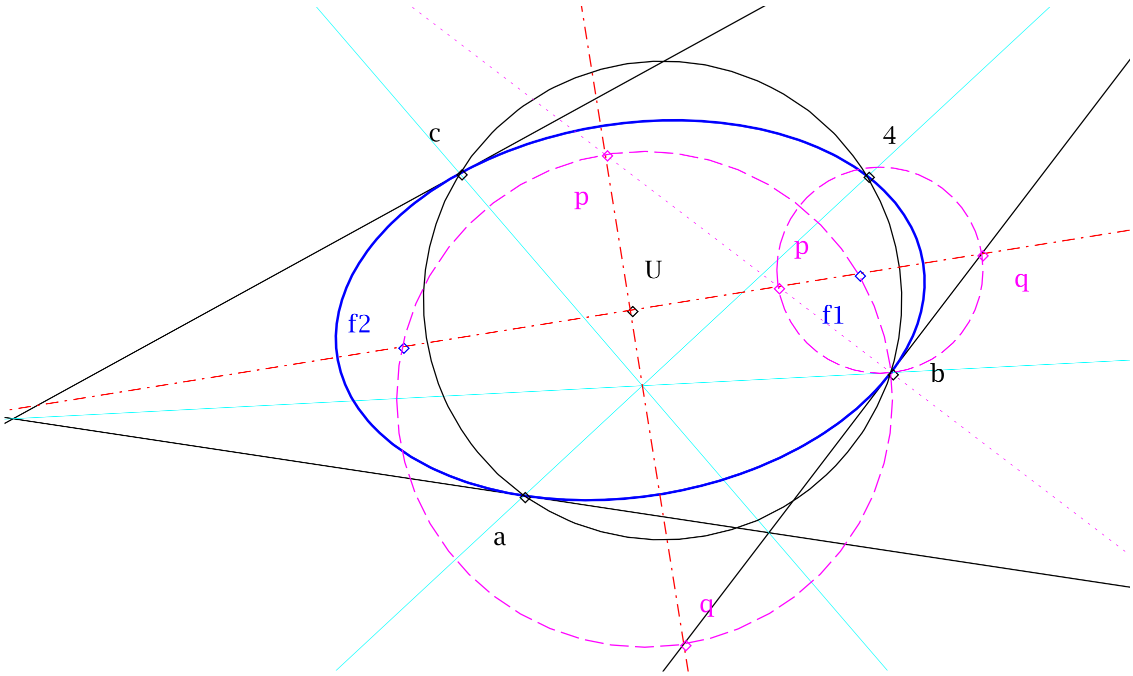


Figure 9.7: The K-ellipse

1. Equation of the K-ellipse is :
$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} - 2\,\frac{xy}{a^2b^2} - 2\,\frac{yz}{b^2c^2} - 2\,\frac{zx}{a^2c^2} = 0$$
perspector is X(6)$= a^2 : b^2 : c^2$, center U is X(39) $= a^2\left(b^2+c^2\right)$, etc.

2. Draw the circumcircle of the contact points $A_K B_K C_K$ and obtain :
$$a^2yz + b^2xz + xyc^2 - (x+y+z)\,\frac{\sum xb^2c^2\left(b^4+c^4+a^2b^2+b^2c^2+a^2c^2-a^4\right)}{2\left(b^2+c^2\right)\left(a^2+c^2\right)\left(a^2+b^2\right)} = 0$$

3. Compute the fourth intersection of this circle with the conic and obtain :
$$Q = a^2\left(b^2-c^2\right)^2\left(a^4+a^2b^2+a^2c^2-b^4-b^2c^2-c^4\right)^2, \text{ etc}$$

4. The axes are the lines through the center that are parallel to the bisectors of $\left(\overbrace{A_K C_K,\, B_K Q}\right)$.
   Therefore, compute :
$$\begin{aligned} T &= \tan\left(\overbrace{A_K C_K,\, A_K B_K}\right) \\ t &= \tan\left(\overbrace{A_K C_K,\, UV}\right) \end{aligned}$$
   where $V \simeq \rho : 1 : -1-\rho$ is an unknown point at infinity, substitute into $T = 2t/\left(1-t^2\right)$ and solve. Solutions are rational, leading to $V_1 = a^2\left(b^2-c^2\right)$, etc =X(512) and $V_2 = a^2\left(a^2b^2+a^2c^2-b^4-c^4\right)$, etc =X(511).

5. Compute the axes as $U \wedge V_1 = \left(a^4-b^2c^2\right) \div a^2$, etc and $U \wedge V_2 = \left(b^2-c^2\right) \div a^2$, etc : the Brocard axis X(3)X(6).
   Having the perspector on an axis is special.

6. The sideline AC and the perpendicular to AC through B_K cut the first axis in P_1, Q_1 and the second in P_2, Q_2. The idea is to draw circle having diameter $[P_1, Q_1]$, then the circle centered at U orthogonal to the former and obtain the focuses by intersection with the axis.

7. More simpler, write $F_i = \mu P_1 + (1-\mu)\, Q_1$ and find μ such that $\left(\overline{UF_i}/\overline{UP_1}\right) \div \left(\overline{UF_i}/\overline{UQ_1}\right) = 1$. These ratios involve vectors that all have the same direction, and no radicals are appearing. In our special case, the equation factors, leading to a well known result (the Brocard points) :
$$F_1 = a^2b^2 : b^2c^2 : c^2a^2 \quad ; \quad F_2 = c^2a^2 : a^2b^2 : b^2c^2$$

8. Proceed the same way with the other axis. Obtain an equation that doesn't factors directly, but whose discriminant splits nevertheless when using the Heron formula (5.5). Finally,
$$F_3,\, F_4 = \left(\begin{array}{c} 4\,\left(a^2b^2+a^2c^2\right) S + i\left(b^4+c^4-a^2c^2-a^2b^2\right) a^2 \\ 4\,\left(b^2c^2+b^2a^2\right) S + i\left(a^4+c^4-a^2b^2-b^2c^2\right) b^2 \\ 4\,\left(c^2a^2+c^2b^2\right) S + i\left(a^4+b^4-a^2c^2-b^2c^2\right) c^2 \end{array}\right)$$

9. To summarize, F_1, F_2 =X(39)±X(511), F_3, F_4 =X(39)±i X(512)/4S. As it should be, the focal distance (from center to a focus) is the same since X(511) and X(512)/4S are obtained by a rotation (in space $\mathcal{V}$).

9.12 Parabola

For the sake of completeness, let us recall the definition.

Definition 9.12.1. A **parabola** is a conic tangent to the infinity line. Two parallel lines make a non proper parabola. The union of line at infinity and another line is ... some kind of circle rather than a "special special" parabola.

Corollary 9.12.2. *The conic defined by matrix* $\boxed{\mathcal{C}}$ *is a parabola when* $\mathcal{L}_\infty \,.\, \mathrm{adj}\left(\boxed{\mathcal{C}}\right) .{}^t\mathcal{L}_\infty = 0$.

9.12.1 Circum parabola

Remark 9.12.3. Let $T_0 = u : v : w$ be the barycentrics of the point at infinity of a circumparabola. Then, from Proposition 9.4.5, its perspector is $P = u^2 : v^2 : w^2$ and lies on the inSteiner ellipse.

Proposition 9.12.4. *Using Morley affixes, let* $\kappa : 0 : 1$ *be the point at infinity of a given circumparabola. Then equation, perspector and focus are:*
$$\boxed{\mathcal{C}_z} \simeq \left(\begin{array}{ccc} 2\,\sigma_3 & -\kappa^2-\sigma_1\sigma_3 & -2\,\sigma_3\kappa \\ -\kappa^2-\sigma_1\sigma_3 & 2\,\sigma_1\kappa^2+4\,\sigma_3\kappa+2\,\sigma_2\sigma_3 & -\sigma_2\kappa^2-\sigma_3^2 \\ -2\,\sigma_3\kappa & -\sigma_2\kappa^2-\sigma_3^2 & 2\,\sigma_3\kappa^2 \end{array}\right)$$

$$P \simeq \begin{pmatrix} \left(\dfrac{3\,\sigma_2 - 4\,\sigma_1^2}{\sigma_3} + \dfrac{\sigma_1\sigma_2^2}{\sigma_3^2}\right)\kappa + \dfrac{4\,\sigma_2^2}{\sigma_3} - 12\,\sigma_1 + (\sigma_2\sigma_1 - 9\,\sigma_3)\,\dfrac{1}{\kappa} \\ \left(\dfrac{2\,\sigma_2^2}{\sigma_3^2} - 6\,\dfrac{\sigma_1}{\sigma_3}\right)\kappa + 2\,\dfrac{\sigma_2\sigma_1}{\sigma_3} - 18 + \left(2\,\sigma_1^2 - 6\,\sigma_2\right)\dfrac{1}{\kappa} \\ \left(\dfrac{\sigma_2\sigma_1}{\sigma_3^2} - \dfrac{9}{\sigma_3}\right)\kappa + \dfrac{4\,\sigma_1^2 - 12\,\sigma_2}{\sigma_3} + \left(3\,\sigma_1 + \dfrac{\sigma_1^2\sigma_2 - 4\,\sigma_2^2}{\sigma_3}\right)\dfrac{1}{\kappa} \end{pmatrix}$$

$$F \simeq \begin{pmatrix} \left(\dfrac{4\,\sigma_1}{\sigma_3} - \dfrac{\sigma_2^2}{\sigma_3^2}\right)\kappa^2 + 8\,\kappa + 2\,\sigma_2 - \dfrac{\sigma_3^2}{\kappa^2} \\ 4\left(\dfrac{1}{\sigma_3}\,\kappa^2 + \dfrac{\sigma_2}{\sigma_3}\,\kappa + \sigma_1 + \sigma_3\,\dfrac{1}{\kappa}\right) \\ \dfrac{-1}{\sigma_3^2}\,\kappa^3 + \dfrac{2\,\sigma_1}{\sigma_3}\,\kappa + 8 - \left(4\,\sigma_2 - \sigma_1^2\right)\dfrac{1}{\kappa} \end{pmatrix}$$

Proof. Use point $\kappa + i\,\kappa\,h$ with $h \to 0$ as the fifth point of the conic and compute the determinant. Therefater, compute the polar triangle and its perspector. Finalize by writting that $\Omega \pm F$ are tangent to the conic. □

Proposition 9.12.5. *The locus of the foci of all the circumscribed parabola is a circular quintic. Singular focus (not on the curve) is X(143), the nine points center of the orthic triangle. Other asymptotes are through points whose barycentrics are respectively,* $2:1:1$, $1:2:1$, $1:1:2$. *Its equation is :*

$$\begin{aligned}
&1024\,\sigma_3^3\,\mathbf{Z}\overline{\mathcal{Z}}\,\left(\mathbf{Z} + \beta\gamma\,\overline{\mathcal{Z}}\right)\left(\mathbf{Z} + \gamma\alpha\,\overline{\mathcal{Z}}\right)\left(\mathbf{Z} + \alpha\beta\,\overline{\mathcal{Z}}\right) \\
&-256\begin{pmatrix} \left(4\,\sigma_2 - \sigma_1^2\right)\sigma_3^2\mathbf{Z}^4 + \left(4\,\sigma_2^2 + 9\,\sigma_1\sigma_3 - \sigma_1^2\sigma_2\right)\sigma_3^2\mathbf{Z}^3\overline{\mathcal{Z}} \\ + \left(3\,\sigma_3^2 + 13\,\sigma_1\sigma_2\sigma_3 - \sigma_1^3\sigma_3 - \sigma_2^3\right)\sigma_3^2\mathbf{Z}^2\overline{\mathcal{Z}}^2 \\ + \left(4\,\sigma_1^2\sigma_3 + 9\,\sigma_2\sigma_3 - \sigma_1\sigma_2^2\right)\sigma_3^3\mathbf{Z}\overline{\mathcal{Z}}^3 + \sigma_3\left(4\,\sigma_1\sigma_3 - \sigma_2^2\right)\overline{\mathcal{Z}}^4 \end{pmatrix}\mathbf{T} \\
&+64\begin{pmatrix} 512\,\sigma_3^2\,\left(\left(\sigma_3 + 4\,\sigma_2\sigma_1 - \sigma_1^3\right)\mathbf{Z}^3 + \left(\sigma_3^3 + 4\,\sigma_1\sigma_2\sigma_3^2 - \sigma_2^3\sigma_3\right)\overline{\mathcal{Z}}^3\right) \\ + \left(\sigma_1^3\sigma_2^2 - 4\,\sigma_1^4\sigma_3 - 8\,\sigma_1\sigma_2^3 + 30\,\sigma_1^2\sigma_2\sigma_3 + 28\,\sigma_2^2\sigma_3 + 33\,\sigma_1\sigma_3^2\right)\sigma_3^2\mathbf{Z}\overline{\mathcal{Z}}^2 \\ + \left(\sigma_1^2\sigma_2^3 - 4\,\sigma_2^4 - 8\,\sigma_1^3\sigma_2\sigma_3 + 30\,\sigma_1\sigma_2^2\sigma_3 + 28\,\sigma_1^2\sigma_3^2 + 33\,\sigma_2\sigma_3^2\right)\sigma_3\mathbf{Z}^2\overline{\mathcal{Z}} \end{pmatrix}\mathbf{T}^2 \\
&+16\,\sigma_3\begin{pmatrix} 2\,\left(-\sigma_1^3\sigma_2^2 + 4\,\sigma_1\sigma_2^3 + 12\,\sigma_1^4\sigma_3 - 42\,\sigma_1^2\sigma_2\sigma_3 - 24\,\sigma_2^2\sigma_3 - 29\,\sigma_1\sigma_3^2\right)\mathbf{Z}^2 \\ \begin{pmatrix} 12\,\sigma_1\sigma_2^4 - 3\,\sigma_1^3\sigma_2^3 + \left(12\,\sigma_1^4\sigma_2 - 31\,\sigma_1^2\sigma_2^2 - 28\,\sigma_2^3\right)\sigma_3 \\ - \left(28\,\sigma_1^3 + 177\,\sigma_1\sigma_2\right)\sigma_3^2 - 77\,\sigma_3^3 \end{pmatrix}\mathbf{Z}\overline{\mathcal{Z}} \\ 2\,\left(-\sigma_1^2\sigma_2^3 + 4\,\sigma_1^3\sigma_2\sigma_3 + 12\,\sigma_2^4 - 42\,\sigma_1\sigma_2^2\sigma_3 - 24\,\sigma_1^2\sigma_3^2 - 29\,\sigma_2\sigma_3^2\right)\sigma_3\overline{\mathcal{Z}}^2 \end{pmatrix}\mathbf{T}^3 \\
&+32\,\sigma_3\begin{pmatrix} \left(\sigma_1^2\sigma_2^4 - 5\,\sigma_1^3\sigma_2^2\sigma_3 + 4\,\sigma_1^4\sigma_3^2 - 4\,\sigma_2^5 + 14\,\sigma_1\sigma_2^3\sigma_3 + 6\,\sigma_1^2\sigma_2\sigma_3^2 + 13\,\sigma_2^2\sigma_3^2 + 35\,\sigma_1\sigma_3^3\right)\overline{\mathcal{Z}} \\ + \left(\sigma_1^4\sigma_2^2 - 5\,\sigma_1^2\sigma_2^3 + 4\,\sigma_2^4 - 4\,\sigma_1^5\sigma_3 + 14\,\sigma_1^3\sigma_2\sigma_3 + 6\,\sigma_1\sigma_2^2\sigma_3 + 13\,\sigma_1^2\sigma_3^2 + 35\,\sigma_2\sigma_3^2\right)\mathbf{Z} \end{pmatrix}\mathbf{T}^4 \\
&+\begin{pmatrix} \sigma_1^4\sigma_2^4 - 8\,\sigma_1^2\sigma_2^2\,\left(\sigma_2^3 + \sigma_1^3\sigma_3\right) + 16\,\left(\sigma_2^6 + \sigma_1^6\sigma_3^2\right) - 80\,\left(\sigma_1^4\sigma_2\sigma_3 + \sigma_1\sigma_2^4\right)\sigma_3 \\ +52\,\left(\sigma_1^3\sigma_2^3 + 2\,\sigma_1^2\sigma_2^2\sigma_3 - 4\,\sigma_1\sigma_2\,\sigma_3^2\right)\sigma_3 - 104\,\left(\sigma_2^3 + \sigma_1^3\sigma_3\right)\sigma_3^2 - 343\,\sigma_3^4 \end{pmatrix}\mathbf{T}^5
\end{aligned}$$

Proof. Elimination is straightforward. The real asymptotes are parallel to the sidelines. □

9.12.2 Inscribed parabola

Remark 9.12.6. (Steiner theorems). From Proposition 9.9.3, the focus F of an inparabola is the isogonal conjugate of its point at infinity U (and is therefore on the circumcircle), while the perspector P is the isotomic conjugate of U (and is therefore on the Steiner circumconic).

9.13 Hyperbola

Definition 9.13.1. An **hyperbola** is a conic that intersects the line at infinity in two different points. An ellipse is a special hyperbola (the intersection points are not real) and a parabola is not an hyperbola.

Proposition 9.13.2. *Let* $\Delta_1 \simeq (\rho, \sigma, \tau)$ *and* $\Delta_2 \simeq (u, v, w)$ *be the asymptotes of an hyperbola* $\mathcal{C}$. *Then equation of* $\mathcal{C}$ *can be written as :*

$$k\,(x + y + z)^2 + (\rho x + \sigma y + \tau z)\,(ux + vy + wz) = 0 \tag{9.4}$$

Proof. Consider the line $\Delta_1 \simeq (\rho, \sigma, \tau)$ and its point at infinity $T_1 = \sigma - \tau : \tau - \rho : \rho - \sigma$. The matrix of the quadratic form is :

$$\boxed{\mathcal{C}} = \frac{1}{2}\left({}^t\Delta_1 \,.\, \Delta_2 + {}^t\Delta_2 \,.\, \Delta_1\right) + k\left(\mathcal{L}_\infty \,.{}^t\mathcal{L}_\infty\right) \tag{9.5}$$

It can be seen that ${}^tT_1 \,.\, \boxed{\mathcal{C}} \,.\, T_1 = 0$ (T_1 belongs to conic) while ${}^tT_1 .\boxed{\mathcal{C}} = \Delta_1$ (the tangent to the conic at T_1 is line Δ_1). Another method is $\Delta_1 \,.\, \mathrm{adj}\left(\boxed{\mathcal{C}}\right) .{}^t\Delta_1 = 0$ (line Δ_1 is tangent to the conic) while $\boxed{\mathcal{C}} .{}^t\Delta_1 = T_1$ (the contact point of Δ is T). □

Corollary 9.13.3. *Equation (9.4) is the parameterization in k of the pencil of hyperbola that share a given pair of asymptotes.*

Proposition 9.13.4. *A rectangular hyperbola is an hyperbola with orthogonal asymptotes. Such an RH is characterized among all the conics by :*

$$\mathrm{trace}\left(\boxed{\mathcal{C}} \,.\, \boxed{\mathcal{M}}\right) = 0$$

Proof. We have $\mathrm{trace}\left({}^t\Delta_1 \,.\, \Delta_2 \,.\, Q\right) = \Delta_2 \,.\, Q \,.{}^t\Delta_1$ as soon as matrix Q is symmetric. Therefore $\mathrm{trace}\left(\boxed{\mathcal{C}} \,.\, \boxed{\mathcal{M}}\right)$ equals $\Delta_2 \,.\, \boxed{\mathcal{M}} \,.{}^t\Delta_1$ and the result follows. □

9.13.1 circumhyperbolas

Proposition 9.13.5. *A circumconic $\mathcal{C}$ can be characterized by one asymptote $\Delta_1 \simeq (\rho, \sigma, \tau)$. Then the second asymptote Δ_2 is $k/\rho : k/\sigma : k/\tau$ where k is the constant appearing in (9.4). Perspector P, center U, contacts at infinity T_1, T_2 are given by :*

$$\begin{array}{ccccccc} P & = & \rho\left(\sigma-\tau\right)^2 & : & \sigma\left(\tau-\rho\right)^2 & : & \tau\left(\rho-\sigma\right)^2 \\ C & = & \rho\left(\sigma^2-\tau^2\right) & : & \sigma\left(\tau^2-\rho^2\right) & : & \tau\left(\rho^2-\sigma^2\right) \\ T_1 & = & \sigma-\tau & : & \tau-\rho & : & \rho-\sigma \\ T_2 & = & \rho\left(\sigma-\tau\right) & : & \sigma\left(\tau-\rho\right) & : & \tau\left(\rho-\sigma\right) \end{array}$$

while the equation of the conic can be rewritten into $T_1 *_b T_2 \div_b X \in \mathcal{L}_\infty$, and asymptotes as $T_1 *_b X \div_b T_2 \in \mathcal{L}_\infty$ and $T_2 *_b X \div_b T_1 \in \mathcal{L}_\infty$. Moreover, $C = T_1 *_b T_2 *_b \mathit{crosssumbar}\left(T_1, T_2\right)$.

Proof. Direct examination. □

Proposition 9.13.6. *Consider a circumconic and its perspector P. The points at infinity are given by :*

$$\left(\begin{array}{c} \left(q^2+r^2-pq-pr\right)p + p\left(q-r\right)\,IST \\ \left(r^2+p^2-qr-qp\right)q + q\left(r-p\right)\,IST \\ \left(p^2+q^2-rp-qr\right)r + r\left(p-q\right)\,IST \end{array}\right)$$

where $IST^2 = p^2+q^2+r^2-2pq-2qr-2rp$ is the equation of the Steiner in-ellipse.

Proof. Direct inspection. □

Proposition 9.13.7. *Any rectangular circumhyperbola can be written as :*

$$-S_a x\left(S_b y - z S_c\right)\mu + S_b y\left(S_a x - z S_c\right) = 0$$

Its perspector is $P = S_b S_c : -S_c S_a \mu : S_b S_a\left(\mu - 1\right)$, on the tripolar of X(4), i.e. on the so-called orthic axis. And its center $C = \mathit{cevadiv}\left(X_2, P\right)$ is on the nine points circle. Kiepert RH is $\mu = \left(a^2-c^2\right)S_b \div \left(b^2-c^2\right)S_a$.

Proof. A circumscribed RH belong to the pencil generated by $BC \cup AH$ and $AC \cup BH$. □

9.13.2 inscribed hyperbolas

Proposition 9.13.8. *An inconic $\mathcal{C}$ can be characterized by one asymptote $\Delta_1 \simeq (\rho, \sigma, \tau)$. Then :*

$$\begin{array}{ccccccc}
\Delta_1 & \simeq & \rho & , & \sigma & , & \tau \\
T_1 & = & \sigma-\tau & : & \tau-\rho & : & \rho-\sigma \\
N_1 & \simeq & \rho\,\sigma+\rho\,\tau-\sigma\,\tau & : & \rho\,\sigma+\sigma\,\tau-\rho\,\tau & : & \rho\,\tau+\sigma\,\tau-\rho\,\sigma \\
\Delta_2 & \simeq & \rho/f & ; & \sigma/g & ; & \tau/h \\
T_2 & = & (\sigma-\tau)\,f^2 & : & (\tau-\rho)\,g^2 & : & (\rho-\sigma)\,h^2 \\
N_2 & \simeq & (\rho\,\sigma+\rho\,\tau-\sigma\,\tau)^{-1} & ; & (\rho\,\sigma+\sigma\,\tau-\rho\,\tau)^{-1} & ; & (\sigma\,\tau+\rho\,\tau-\rho\,\sigma)^{-1} \\
P & = & 1\div\left((\sigma-\tau)\,\rho^2\right) & : & 1\div\left((\tau-\rho)\,\sigma^2\right) & : & 1\div\left((\rho-\sigma)\,\tau^2\right) \\
C & = & (\sigma-\tau)\,f & : & (\tau-\rho)\,g & : & (\rho-\sigma)\,h
\end{array}$$

where $(f, g, h) \simeq N_1 \simeq anticomplem\,(isot\,(\Delta))$ is the Newton line associated with line Δ_1 (cf Proposition **??***). Therefore :*

$$\Delta_2 = \Delta_1 \div_b N_1 \;;\; T_2 = T_1 *_b N_1 *_b N_1 \;;\; N_2 = G \div_b N_1$$
$$C = T_1 *_b N_1 = T_2 *_b N_2 = crossmul\,(G, P) \;;\; P = crossdiv\,(G, C)$$

Proof. Let $\boxed{M}$ be the matrix $\boxed{IC}\,(p:q:r)$ of the general inconic with perspector $p:q:r$. Then formula giving P from Δ is *obtained* by elimination from $\Delta\,.\,\mathrm{adj}\left(\boxed{M}\right).{}^t\Delta = 0$ (tangency) and $\mathcal{L}_\infty\,.\,\mathrm{adj}\left(\boxed{M}\right).{}^t\Delta = 0$ $(T \in \mathcal{L}_\infty)$. Thereafter, all formulae are *proven* by direct computing from matrix $\boxed{\mathcal{C}} = \boxed{IC}\,(P)$ where P is as given (cf. Stothers, 2003a). □

Proposition 9.13.9. *Consider an inconic and its center U. The points at infinity are given by :*

$$\left(\begin{array}{c} u^2\left(v^4+w^4-u^2v^2-u^2w^2\right) \\ v^2\left(u^4+w^4-v^2w^2-u^2v^2\right) \\ w^2\left(u^4+v^4-u^2w^2-v^2w^2\right) \end{array}\right) \pm OST\,(v+w-u)\,(w+u-v)\,(u+v-w) \left(\begin{array}{c} \left(v^2-w^2\right)u^2 \\ \left(w^2-u^2\right)v^2 \\ \left(u^2-v^2\right)w^2 \end{array}\right)$$

where $OST^2 = -qr - rp - pq$ is the equation of the Steiner out-ellipse.

Proof. Direct inspection. □

Proposition 9.13.10. *Consider point $U = v-w : w-u : u-v \in \mathcal{L}_\infty$ and its tripolar $\Delta_0 \simeq [1/(v-w)\,,\, 1/(w-u)\,,\, 1/(u-v)]$. This line is tangent to Steiner in-ellipse and the contact point is $T_0 = (v-w)^2 : (w-u)^2 : (u-v)^2$. Define (index i=inscribed, c=circumscribed) lines $\Delta_i = tripole\,(T_0)$, $\Delta_c \simeq [v-w,\; w-u,\; u-v]$ and point*

$$C = TG\,(U) = (v-w)^2\,(v+w-2u) : (w-u)^2\,(u+w-2v) : (u-v)^2\,(u+v-2w)$$

Then C is the common center of a circum-hyperbola with asymptotes Δ_0, Δ_c and an in-hyperbola with asymptotes Δ_0, Δ_c. The locus of C (and also of the circum-perspector) is K219 :

$$\sum_3 x^3 - \sum_6 x^2y + 3xyz = 0$$

while the locus of the in-perspector is : $\sum_6 x^2y - 6xyz = 0$. All lines Δ_c contain $G = X_2$ while envelope of the Δ_i is cubic : $\sum_3 x^3 + 3\sum_6 x^2y - 21xyz = 0$.

Proof. In all these cubics, G is an isolated point (and don't belong to the locus). Otherwise, computing as usual. □

9.14 Diagonal conics

Triangle ABC is autopolar wrt conic Γ if, and only if, the non-diagonal coefficients vanish. Such a conic is called diagonal.

Proposition 9.14.1. *Let $\Gamma(\mu)$ be a pencil of diagonal conics :*

$$\Gamma(\mu) \doteq (1-\mu)\left(\alpha_1 x^2+\beta_1 y^2+\gamma_1 z^2\right)+\mu * \left(\alpha_2 x^2+\beta_2 y^2+\gamma_2 z^2\right)=0$$

and $U = u : v : w$ a point, not a vertex of ABC. Then polar lines of U wrt all the conics of the pencil are concurring at a point U^* that will be called the isoconjugate of U wrt the pencil. In fact, $U^* = U_P^*$ –cf (14.2)– where :

$$P=\beta_1\gamma_2-\gamma_1\beta_2 : \gamma_1\alpha_2-\alpha_1\gamma_2 : \alpha_1\beta_2-\beta_1\alpha_2$$

The four fixed points (real or not) of the conjugacy, i.e. the points $\pm\sqrt{p} : \pm\sqrt{q} : \pm\sqrt{r}$, are the points common to all conics of the pencil.

When a pair of isoconjugates U_1 and U_2 is known, P is known and therefore the isoconjugacy. The pencil contains the conic Γ_1 through U_1, *cevadiv* (U_2, U_1) and the vertices of their respective anti-cevian triangles. Conic Γ_2 is defined cyclically. Both conics are tangent to U_1U_2.

Circumconic $CC(P)$ is together the P-isoconjugate of $\mathcal{L}_\infty$, the locus of centers of the conics of the pencil and the conic that contains the six midpoints of the quadrangle formed by the four fixed points.

9.15 The bitangent pencil (from 6.4 and 2.4)

Definition 9.15.1. All of the conics that are tangent to two fixed lines at two given points form a pencil, called the bitangent pencil $\mathcal{F}$. We define B, C as the contact points and A as the intersection of the tangents. Two points of interest are called M and A'. Point $M = (B+C)/2$ defines the median AM which is the line of centers. Point A' is the second intersection of the ABC circumcircle and the A-symedian of this triangle.

Theorem 9.15.2. *The punctual and tangential equation of these conics are :*

$$\boxed{\mathcal{C}_b} \simeq \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;;\; \boxed{\mathcal{C}_b^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix} \;;\; \boxed{\mathcal{C}_z} = {}^t\boxed{Lu}^{-1} \cdot \boxed{\mathcal{C}_b} \cdot \boxed{Lu}^{-1} ; \boxed{\mathcal{C}_z^*} = \boxed{Lu} \cdot \boxed{\mathcal{C}_b^*} \cdot {}^t\boxed{Lu}$$

The Morley affixes $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ of a focus of the conic $\mathcal{C}(\lambda) \in \mathcal{F}$ are bound to the parameter λ by :

$$\frac{-1}{2\lambda} = \frac{(\mathbf{Z}-\beta\mathbf{T})(\mathbf{Z}-\gamma\mathbf{T})}{(\mathbf{Z}-\alpha\mathbf{T})^2} = \frac{\left(\overline{\mathcal{Z}}-\mathbf{T}/\beta\right)\left(\overline{\mathcal{Z}}-\mathbf{T}/\gamma\right)}{\left(\overline{\mathcal{Z}}-\mathbf{T}/\alpha\right)^2} \tag{9.6}$$

and the focus is located on the "focal cubic" $\mathcal{K}$:

$$\left(\frac{2}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}\right)\mathbf{Z}^2\overline{\mathcal{Z}}+(\gamma+\beta-2\,\alpha)\,\mathbf{Z}\overline{\mathcal{Z}}^2+\left(\frac{1}{\beta\,\gamma}-\frac{1}{\alpha^2}\right)\mathbf{Z}^2\mathbf{T}+\left(\alpha^2-\beta\,\gamma\right)\overline{\mathcal{Z}}^2\mathbf{T}+$$
$$2\left(\frac{\alpha}{\gamma}+\frac{\alpha}{\beta}-\frac{\beta+\gamma}{\alpha}\right)\mathbf{Z}\overline{\mathcal{Z}}\mathbf{T}+\left(\frac{\beta+\gamma}{\alpha^2}-\frac{2\alpha}{\beta\,\gamma}\right)\mathbf{Z}\mathbf{T}^2+\left(\frac{2\,\beta\gamma}{\alpha}-\frac{\alpha^2}{\gamma}-\frac{\alpha^2}{\beta}\right)\overline{\mathcal{Z}}\mathbf{T}^2+\left(\frac{\alpha^2}{\beta\,\gamma}-\frac{\beta\,\gamma}{\alpha^2}\right)\mathbf{T}^3$$

This curves goes through both umbilics, vertices B, C and two times through A. The asymptote Δ_∞ is the parallel to the median AM through point $\Omega' = (3A - A')/2$

Proof. The Plucker definition gives $(F \wedge \Omega_x) \cdot \boxed{\mathcal{C}_z^*} \cdot {}^t(F \wedge \Omega_x) = 0 = (F \wedge \Omega_y) \cdot \boxed{\mathcal{C}_z^*} \cdot {}^t(F \wedge \Omega_y)$. A separation of the variables occurs, giving one equation in the upper view $\mathbf{Z} : \mathbf{T}$ and another in the lower view $\overline{\mathcal{Z}} : \mathbf{T}$. Ombilical property is obvious. The real asymptote is easily obtained from the gradient. □

Proposition 9.15.3. *The focal cubic of the bitangent pencil can be parameterized/constructed as follows. A variable point $K = \kappa : 1 : 1/\kappa$ on the circumcircle defines a variable line AK. Reflect B, C into AK and obtain B_k, C_k. Then point $F = BC_k \cap CB_k$ is on the cubic.*

Proof. From (9.6), line FA is a bissector of angle FB, FC. We obtain :

$$F(\kappa) \simeq \left(\begin{array}{c} \alpha(\alpha-\beta)(\alpha-\gamma)\kappa^3 - \alpha^2(\alpha\beta+\alpha\gamma-2\beta\gamma)\kappa^2 + \alpha\beta\gamma\left(\alpha^2-\beta\gamma\right)\kappa \\ -\alpha\left(\alpha\beta+\alpha\gamma-2\beta\gamma\right)\kappa^2 + \alpha\beta\gamma\left(2\alpha-\beta-\gamma\right)\kappa \\ \left(\beta\gamma-\alpha^2\right)\kappa^2 + \beta\gamma\left(2\alpha-\beta-\gamma\right)\kappa + \beta\gamma\left(\alpha-\beta\right)\left(\alpha-\gamma\right) \end{array} \right) \qquad \square$$

Proposition 9.15.4. *In this construction, the other visible focus F' of the same conic is obtained by using the line AK' such that lines AK and AK' are equally inclined on lines AB, AC. Moreover, in the upper view $\mathbf{Z}:\mathbf{T}$, the focuses are exchanged by the involutory homography whose fixed points are A and A', the second intersection of the A-symedian with the circumcircle.*

Proof. For the second part, write and factor $\lambda\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)-\lambda\left(z:t:\zeta\right)$ from (9.6). This gives $(z\mathbf{T}-t\mathbf{Z})$ together with another first degree factor with respect to $\mathbf{Z},\mathbf{T}$ and also with respect to z, t. In the upper view $\mathbf{Z}:\mathbf{T}$, this induces the identity together with another homography, whose matrix is

$$\boxed{\psi} \simeq \left(\begin{array}{cc} \beta\gamma-\alpha^2 & \alpha^2\beta+\gamma\alpha^2-2\alpha\beta\gamma \\ \beta+\gamma-2\alpha & \alpha^2-\beta\gamma \end{array} \right)$$

A fixed point is $A=\alpha:1$. The other is $A' \doteq \alpha\beta+\alpha\gamma-2\beta\gamma : 2\alpha-\beta-\gamma$ on the circumcircle. One can check that :

$$\omega_{AA'} \times \omega_{AM} = \frac{\alpha\left(\gamma\alpha+\alpha\beta-2\beta\gamma\right)}{\beta+\gamma-2\alpha} \times \frac{\alpha\beta\gamma\left(\beta+\gamma-2\alpha\right)}{\gamma\alpha+\alpha\beta-2\beta\gamma} = (-\alpha\beta)\times(-\alpha\gamma) = \omega_{AB}\times\omega_{AC}$$

For the first part, write that $f(\nu) \simeq \psi\left(f(\kappa)\right)$ where $f(\kappa)$ is the upper part, i.e. $\mathbf{Z}:\mathbf{T}$ of $F(\kappa)$. This leads to a second degree equation. One solution verifies $\nu\kappa=\beta\gamma$ as required. The other is not a turn (and leads to a non visible focus). Another method is a direct substitution of $F(\kappa)$ into (9.6). This leads to a fourth degree polynomial whose roots are κ, $\beta\gamma/\kappa$ and other two that are not turns (and lead to boths unvisible focuses). $\square$

Proposition 9.15.5. *When center U is chosen on the median AM, the focuses F, F' can be constructed as follows. Draw the bissectors Δ_1, Δ_2 of UA, UA'. Cut them at H_1, H_2 by the perpendicular bisector of $[A, A']$. Draw circle $\gamma_1(H_1, A)$ and cut Δ_2. Additionally, draw circle $\gamma_2(H_2, A')$ and cut Δ_1. This gives the four focuses.*

Proof. This comes from the involutory homography ψ. $\square$

Proposition 9.15.6 (Cissoidal property)**.** *Consider circle γ through A and centered at $\Omega'=(3A-A')/2$. Define P as the second intersection of γ and line AF. Define P' from F'. Then P' is the reflection of P into the asymptote Δ_∞ of the cubic $\mathcal{K}$. Define Q as the intersection of PF with Δ_∞, and also Q' as $P'F'\cap\Delta_\infty$. Then we have the cissoidal property : $\overrightarrow{PQ}=\overrightarrow{AF}$. By the way, intersection Q_0 of Δ_∞ and external bissector of AB, AC belongs to γ. Moreover $\overrightarrow{\Omega'Q}\cdot\overrightarrow{\Omega'Q'}=\left|\overrightarrow{\Omega'Q_0}\right|^2$: points Q, Q' are the reflection of each other into circle γ.*

Proof. Direct computation. In order to deal with with Q_0, we have to use the second degree Lubin frame. $\square$

Proposition 9.15.7. *The isogonal conjugates F^*, F'^* of the focuses of the $\mathcal{C}\in\mathcal{F}$ belong to a circle. A diameter of this circle is provided by the feet J_1, J_2 of the A-bisectors. Points F^*, F'^* are symmetrical with respect to this diameter. Moreover, the point F^* is the image of point Q (see preceding proposition) by a skew similitude whose fixed directions are the bissectors of the angle AM, BC.*

Proof. The κ-degrees of the affixes of F are $3,2,2$. The κ-degrees of the affixes of F^* are $1,1,2$: these points belong at least to a conic. Umbilics belong to the cubic. Their isogonal transforms belong to the conic: we have a circle. Using the second degree Lubin frame, the matrix of the locus is :

$$\boxed{\varphi} \underset{2}{\simeq} \left(\begin{array}{ccc} 0 & 2\alpha^2-\beta^2-\gamma^2 & \beta^2\gamma^2-\alpha^4 \\ 2\alpha^2-\beta^2-\gamma^2 & 2\beta^2\gamma^2-2\alpha^4 & \alpha^2\left(\alpha^2\beta^2+\alpha^2\gamma^2-2\beta^2\gamma^2\right) \\ \beta^2\gamma^2-\alpha^4 & \alpha^2\left(\alpha^2\beta^2+\alpha^2\gamma^2-2\beta^2\gamma^2\right) & 0 \end{array} \right)$$

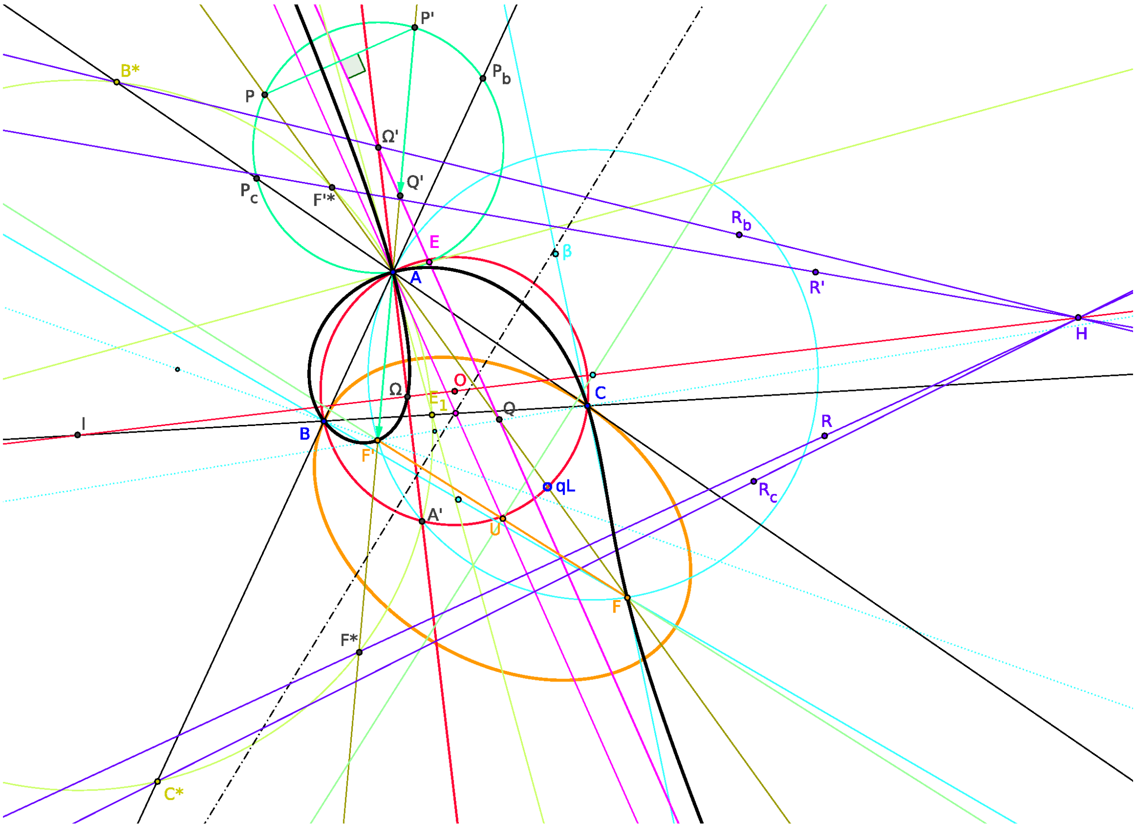


Figure 9.8: Two transformations of the focal cubic.

Its intersections with line BC are

$$J_1 \underset{2}{\simeq} \left(\begin{array}{c} \left(\alpha^2\beta\,\gamma + \alpha^2\beta^2 + \alpha^2\gamma^2 - \beta^2\gamma^2\right)\gamma\,\beta \\ \left(\alpha^2 + \beta\,\gamma\right)\gamma\,\beta \\ \beta^2 + \beta\,\gamma + \gamma^2 - \alpha^2 \end{array} \right)$$

and J_2 obtained by $\beta\gamma \mapsto -\beta\gamma$. Its center is $BC \cap O\Omega$, where $O =$X(3).

We search for a matrix $\boxed{Q}$ such that $F^* \simeq \boxed{Q}\,.\,P$. In order to success into identifying with respect to powers of κ, we have to normalize both P and F^*. We obtain :

$$\boxed{Q} \simeq \left(\begin{array}{ccc} 0 & Q_o & \dfrac{(\beta+\gamma-2\,\alpha)\,\alpha\beta\gamma}{\alpha^2-\beta\,\gamma} \\ 0 & 1 & 0 \\ \dfrac{\alpha\beta+\alpha\gamma-2\,\beta\gamma}{\beta\gamma\,(\alpha^2-\beta\,\gamma)} & \overline{Q}_o & 0 \end{array} \right) \quad \text{where}$$

$$Q_o = \frac{\beta^2+\gamma^2+4\,\beta\,\gamma}{\beta+\gamma} + \frac{3\beta\gamma\,(\beta+\gamma-2\,\alpha)}{\alpha^2-\beta\,\gamma} + \frac{\beta\gamma\,(\beta-\gamma)^2}{(2\,\beta\gamma-\alpha\beta-\alpha\gamma)\,(\beta+\gamma)}$$

umbilics are exchanged and the conclusion follows. □

9.16 The Miquel pencil of conics

Definition 9.16.1. All of the conics that are tangent to four fixed lines form a pencil, called the Miquel pencil $\mathcal{F}$ of these four lines.

Lemma 9.16.2. *Let us use describe our four lines as the three sidelines of ABC and the transversal whose tripole is $p : q : r$. See Theorem 3.4.4 for more details. In the Lubin frame, the transversal,*

the Newton line, the Miquel point Ω of the quadrilateral and the homography ψ are :

$$A'_z,\, B'_z,\, C'_z \simeq \left(\begin{array}{c} \beta\, q-\gamma\, r \\ q-r \\ \dfrac{q}{\beta}-\dfrac{r}{\gamma} \end{array}\right),\left(\begin{array}{c} \alpha\, p-\gamma\, r \\ p-r \\ \dfrac{p}{\alpha}-\dfrac{r}{\gamma} \end{array}\right),\left(\begin{array}{c} \beta\, q-\alpha\, p \\ q-p \\ \dfrac{q}{\beta}-\dfrac{p}{\alpha} \end{array}\right)$$

$$Newton \simeq {}^{t}\left[\begin{array}{c} \alpha\,(\gamma-\beta)\,qr+\beta\,(\alpha-\gamma)\,rp+\gamma\,(\beta-\alpha)\,pq \\ \alpha\left(\beta^{2}-\gamma^{2}\right)qr+\beta\left(\gamma^{2}-\alpha^{2}\right)rp+\gamma\left(\alpha^{2}-\beta^{2}\right)pq \\ \left((\gamma-\beta)\;qr+(\alpha-\gamma)\;rp+(\beta-\alpha)\;pq\right)\alpha\beta\gamma \end{array}\right]$$

$$\Omega \simeq \left(\begin{array}{c} c_1/c_0 \\ 1 \\ c_0/c_1 \end{array}\right),\ \boxed{\psi}=\left(\begin{array}{ccc} c_1 & -c_2 & 0 \\ c_0 & -c_1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\text{where}\begin{cases} c_2 & =\alpha^{2}\,(\gamma-\beta)\,p+\beta^{2}\,(\alpha-\gamma)\,q+\gamma^{2}\,(\beta-\alpha)\,r \\ c_1 & =\alpha\;(\gamma-\beta)\,p+\beta\;(\alpha-\gamma)\,q+\gamma\;(\beta-\alpha)\,r \\ c_0 & =(\gamma-\beta)\,p+(\alpha-\gamma)\,q+(\beta-\alpha)\,r \\ c_{-1} & =\frac{1}{\alpha}\;(\gamma-\beta)\,p+\frac{1}{\beta}\;(\alpha-\gamma)\,q+\frac{1}{\gamma}\;(\beta-\alpha)\,r \end{cases}$$

Remark 9.16.3. Quantity c_{-1} is only for cosmetic purposes. In fact, we have the bounding relation :

$$c_2-c_1\sigma_1+c_0\sigma_2-c_{-1}\sigma_3=0$$

Proposition 9.16.4. *Fixed points of the isoconjugacy Ψ generated by ψ are given by :*

$$\Phi \simeq \left(\begin{array}{c} c_1/c_0+W_u/c_0 \\ 1 \\ c_0/c_1+\alpha\beta\gamma\,W_d/c_1 \end{array}\right)$$

where the up and down radicals are given by :

$$\begin{array}{rcl} W_u^{2} & = & (\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)\times\left(qr\,(\gamma-\beta)+rp\,(\alpha-\gamma)+pq\,(\beta-\alpha)\right) \\ W_d^{2} & = & \dfrac{(\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)}{\alpha^{2}\beta^{2}\gamma^{2}}\times\left(qr\,\dfrac{\gamma-\beta}{\beta\,\gamma}+rp\,\dfrac{\alpha-\gamma}{\alpha\,\gamma}+pq\dfrac{\beta-\alpha}{\alpha\,\beta}\right) \end{array}$$

They can be constructed as follows. Lines Δ_1, Δ_2 are the common bissectors of $\Omega A, \Omega A'$, $\Omega B, \Omega B'$, $\Omega C, \Omega C'$ and δ_A is the perpendicular bissector of $[A, A']$. Then $H_1 = \Delta_1 \cap \delta_A$ (resp. $H_2 = \Delta_2 \cap \delta_A$) is the center of a circle by A, A' that cuts Δ_2 (resp Δ_1) at the four Φ_j points.

Proof. These points are characterized by

$$c_0\mathbf{Z}^{2}-2\,c_1\,\mathbf{ZT}+c_2\mathbf{T}^{2}=0\quad;\quad c_1\overline{\mathcal{Z}}^{2}-2\,c_0\overline{\mathcal{Z}}\mathbf{T}+c_{-1}\,\mathbf{T}^{2}=0$$

□

Proposition 9.16.5 (Newton)**.** *All the conics that are tangent to four given lines have their centers on another same line, that goes through the midpoints of the diagonal pairs AA', BB', CC' (the Newton line). When $U \in New$ is defined as $K\, M_b + (1-K)\, M_c$ the conic can be written as :*

$$\boxed{\mathcal{C}_b^{*}}\simeq(r-p)\left(\begin{array}{ccc} 0 & 0 & -p \\ 0 & 0 & q \\ -p & q & 0 \end{array}\right)+K\,p\left(\begin{array}{ccc} 0 & p-q & r-p \\ p-q & 0 & q-r \\ r-p & q-r & 0 \end{array}\right)$$

The first part is the tangential conic "all lines through C' or C", the second part is a non degenerated parabola (with same direction as the Newton line).

Proof. Let $\rho : \sigma : \tau$ be the "service point" of a conic $\mathcal{C} \in \mathcal{F}$. We have $\boxed{\mathcal{C}_b^{*}} = [0, \tau, \sigma; \tau, 0, \rho; \sigma, \rho, 0]$ and therefore $p\rho + q\sigma + r\tau = 0$. Since $U = \sigma + \tau : \tau + \rho : \rho + \sigma$, the Newton line is :

$$[q+r-p,\; r+p-q,\; p+q-r]$$

and the conclusions follow (remember that $p : q : r$ is the tripole of the transversal). □

Theorem 9.16.6. *The Morley affixes* $\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}$ *of a focus of the conic* $\mathcal{C}(K)\in\mathcal{F}$ *are bound to the parameter* K *by :*

$$K=\frac{(q-p)(\mathbf{Z}-\beta\,\mathbf{T})((r-p)\,\mathbf{Z}-(\gamma\,r-\alpha\,p)\,\mathbf{T})}{p\,(\mathbf{Z}c_0-\mathbf{T}c_1)\,\mathbf{T}} \tag{9.7}$$

and the focus is located on the "focal cubic" $\mathcal{K}$ *:*

$$\frac{c_1}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}}+c_0\,\mathbf{Z}\overline{\mathcal{Z}}^2-\frac{c_0}{\sigma_3}\,\mathbf{T}\mathbf{Z}^2-\frac{c_2+c_0\sigma_2}{\sigma_3}\,\mathbf{T}\mathbf{Z}\overline{\mathcal{Z}}-c_1\mathbf{T}\overline{\mathcal{Z}}^2+\frac{c_0\sigma_1}{\sigma_3}\,T^2\mathbf{Z}+\frac{c_1\sigma_2}{\sigma_3}\,\mathbf{T}^2\overline{\mathcal{Z}}+\frac{c_2-c_1s_1}{s_3}\,\mathbf{T}^3$$

Proof. Matrix $\boxed{\mathcal{C}_z^*}$ is obtained as ${}^t\boxed{Lu}\cdot\boxed{\mathcal{C}_b^*}\cdot\boxed{Lu}$ and then Plucker method is used. Some factors $(q-r)$ are appearing during the elimination process, but not all the $(p-q)(q-r)(r-p)$. Nothing special occurs when P is on a median (but not at the centroid). □

Theorem 9.16.7. *The focuses* F_j *of a given conic* $\mathcal{C}(K)\in\mathcal{F}$ *are exchanged by homographies* $\psi,\overline{\psi}$. *They can be constructed as follows. Call* $New^{\perp}$ *the perpendicular to the Newton line at* Ω. *Draw the bissectors* Δ_1, Δ_2 *of* $U\Phi_1$, $U\Phi_2$. *Cut them at* H_1, H_2 *by* $New^{\perp}$. *Draw circle* $\gamma_1(H_1,\Phi_1)$ *and cut* Δ_2. *Additionally, draw circle* $\gamma_2(H_2,\Phi_2)$ *and cut* Δ_1.

Proof. Write and factor $K\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)-K(z:t:\zeta)$ from (9.7). This gives $(p-q)(p-r)$ but not $(q-r)$, being smoth except at the centroid (i.e. when the fourth line at infinity). Otherwise, this gives $(z\mathbf{T}-t\mathbf{Z})$ together with another first degree factor with respect to $\mathbf{Z},\mathbf{T}$ and also with respect to z,t. In the upper view $\mathbf{Z}:\mathbf{T}$, this induces the identity together with another homography. Since the later has to provide $A\longleftrightarrow A'$, $B\longleftrightarrow B'$, $C\longleftrightarrow C'$, it has to be ψ. The same occurs in the lower view $\overline{\mathcal{Z}}:\mathbf{T}$, and the conclusion follows. □

Remark 9.16.8. In the Geogebra Figure 9.9, the orange conic is drawn as follows. Reflect focuses F,F' into sideline AC and obtain F_b,F_b'. Then point $E_b=FF_b'\cap F'F_b$ on sideline AC belongs to the conic. In the same way, obtain the point E_c on AB. We draw both conics, an ellipse and an hyperbola, with focuses F,F' that go through E_b, but only the one that goes also through E_c is displayed.

Proposition 9.16.9. *When the transversal is tangent to* one *of the inexcircles of triangle* ABC, *the pencil contains* one *circle and the focal conic has a double point. This point is the center of the circle (see Figure 9.9b). When the transversal touches* two *of the inexcircles, it degenerates into the Newton line and a circle having the corresponding inexcenters as antipodal points.*

Proof. Only circles have equal focuses. □

9.17 Tg and Gt mappings

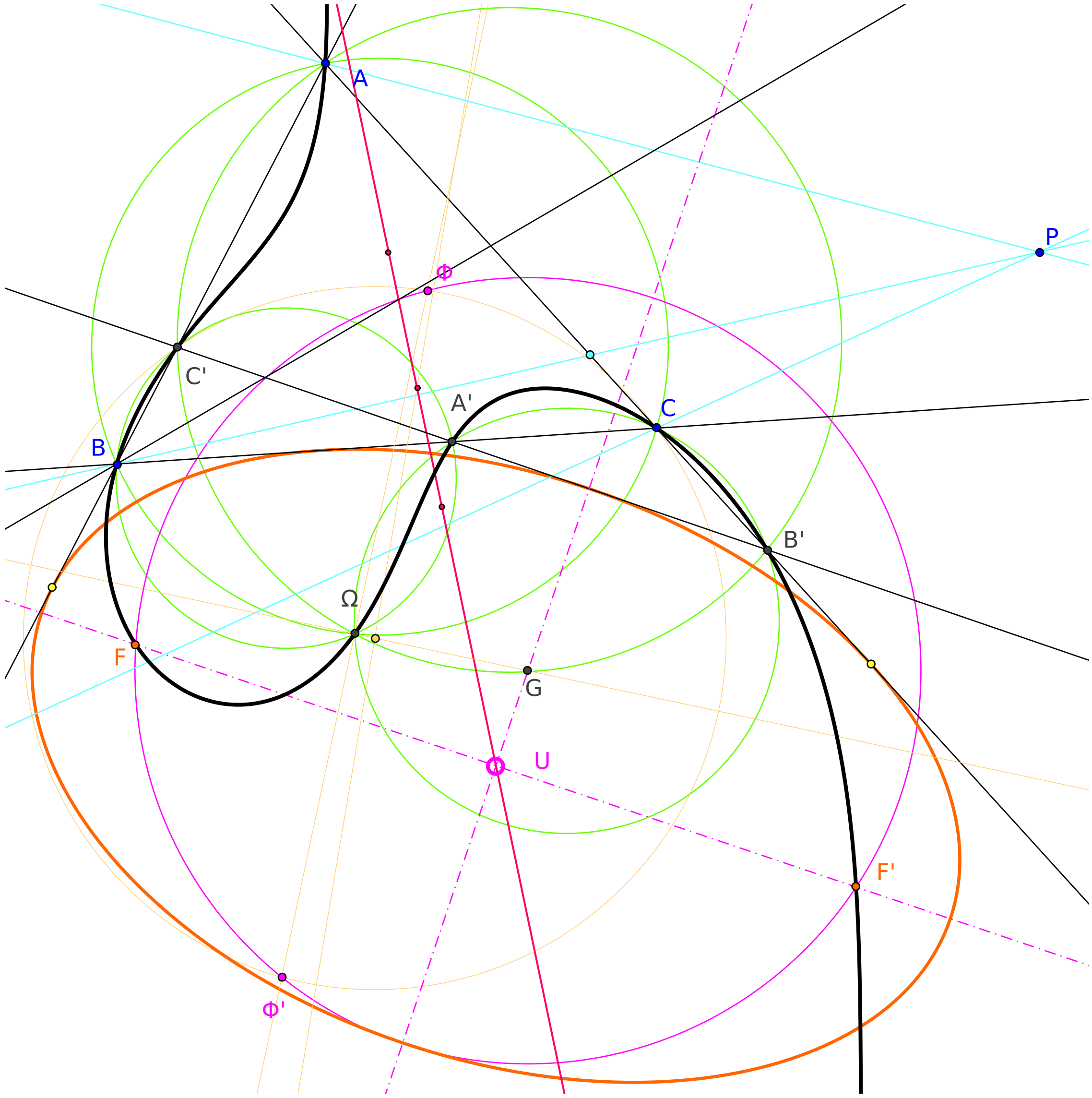
Definition 9.17.1. Tg and Gt mappings. Suppose U is a point not on a sideline of ABC. Let :
$gU=$ isogonal conjugate of U, $tU=$ isotomic conjugate of U
$tgU=$ isotomic conjugate of gX, $gtU=$ isogonal conjugate of tU
$GtU=$ intersection of lines $U-tU$ and $gU-gtU$,
$TgU=$ intersection of lines $U-gU$ and $tU-tgU$.
If $U=u:v:w$ (barycentrics), then :

$$GtU=\frac{a^2\left(b^2-c^2\right)}{(v^2-w^2)\,u}:\frac{b^2\left(c^2-a^2\right)}{(w^2-u^2)\,v}:\frac{c^2\left(a^2-b^2\right)}{(u^2-v^2)\,w}$$

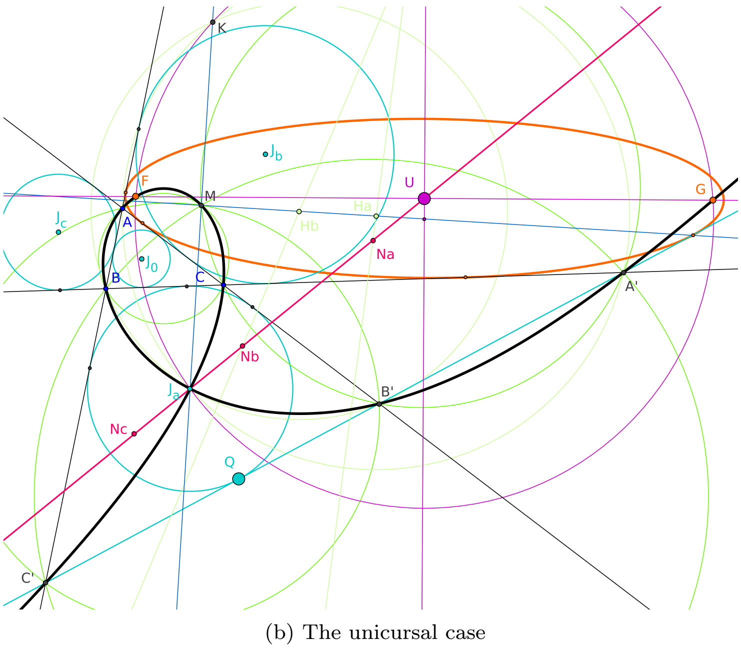
$$TgU=\frac{b^2-c^2}{(w^2b^2-v^2c^2)\,u}:\frac{c^2-a^2}{(u^2c^2-w^2a^2)\,v}:\frac{a^2-b^2}{(v^2a^2-u^2b^2)\,w}$$

Proposition 9.17.2. *For any point* U, *not on a sideline of* ABC, *points* A, B, C, gU, tU, TgU, GtU *are on a same conic (Tuan, 2006). The perspector of this conic is* $X_{512}\div_b U$. *This conic is the isogonal image of line* $U-gtU$ *and also the isotomic image of line* $U-tgU$.

Proof. Straightforward computation. □



(a) The general case



(b) The unicursal case

Figure 9.9: The Miguel pencil and its focal cubic

Example 9.17.3. Points X(3112) to X(3118) are related to Gt and Tg functions.
The X(31)-conic passes through X(I) for

I = 75, 92, 313, 321, 561, 1441, 1821, 1934, 2995, 2997, 3112, 3113

The X(32)-conic passes through X(I) for

I = 76, 264, 276, 290, 300, 301, 308, 313, 327, 349, 1502, 2367, 3114, 3115

The X(76)-conic passes through X(I) for

I = 6, 32, 83, 213, 729, 981, 1918, 1974, 2207, 2281, 2422, 3114, 3224, 3225

Chapter 10

More about circles

10.1 General results

Let us start by recalling two key results.

Theorem 10.1.1 (Already stated in Section 5.3 as Theorem 5.3.2)**.** *Let Ω be the circle centered at P with radius ω. The* ***power formula*** *giving the Ω-power of any point $X = x : y : z$ from the power at the three vertices of the reference triangle is :*

$$\begin{aligned} power\,(\Omega,\, X) &\doteq |PX|^2 - \omega^2 = \frac{ux + vy + wz}{x + y + z} - \frac{a^2yz + b^2xz + c^2xy}{(x + y + z)^2} \\ \text{where } u &= power\,(\Omega,\, A)\,, \text{ etc} \end{aligned} \tag{10.1}$$

Definition 10.1.2 (Already stated in Section 5.3, Proposition 5.3.3)**.** From $0 = power\,(\Gamma,\, A)\,$, etc, we can define the standard equation of the circumcircle as :

$$\Gamma\,(x, y, z) \doteq a^2yz + b^2xz + c^2xy = 0 \tag{10.2}$$

Proposition 10.1.3. *Let $\mathcal{C}$ be a conic, with matrix $\boxed{\mathcal{C}} = (m_{jk})$ (notations of Definition 9.3.1) . Then $\mathcal{C}$ is a cycle if and only if, for a suitable factor k, we have :*

$$\boxed{\mathcal{C}} - \frac{1}{2}\left(\begin{array}{ccc} 2\,m_{11} & m_{11} + m_{22} & m_{33} + m_{11} \\ m_{11} + m_{22} & 2\,m_{22} & m_{22} + m_{33} \\ m_{33} + m_{11} & m_{22} + m_{33} & 2\,m_{33} \end{array}\right) = k\left(\begin{array}{ccc} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{array}\right)$$

Proof. Obvious from (10.1). $\square$

Proposition 10.1.4. *Four points at finite distance belong to the same cycle (aka circle or straight line) when their barycentrics $p_i : q_i : r_i$ are such that :*

$$\det_{i=1}^{i=4}\left[\frac{\Gamma\,(p_i,\, q_i,\, r_i)}{p_i + q_i + r_i},\, p_i,\, q_i,\, r_i\right] = 0 \tag{10.3}$$

Proof. Obvious from (10.1). $\square$

Computed Proof. Denominators are a reminder of the fact that circles don't escape to infinity. Write the Cartesian equation of the circle as :

$$\Delta_{cart} \doteq \det_{i=1}^{i=4}\left[\xi_i^2 + \eta_i^2,\, \xi_i,\, \eta_i, 1\right] = 0$$

where ξ, η are the Cartesian coordinates of the points. Substitute these coordinates by :

$$\xi = \frac{x\xi_a + y\xi_b + z\xi_c}{x + y + z},\, \eta = \frac{x\eta_a + y\eta_b + z\eta_c}{x + y + z}$$

and obtain another determinant $\Delta'(x,y,z)$. Then compute $F\,.\,\Delta'\,.\,T^{-1}\,.\,G$ where F is the diagonal matrix $diag\,(p_i+q_i+r_i)$ and

$$T=\left[\begin{array}{cccc}1&0&0&0\\0&\xi_a&\eta_a&1\\0&\xi_b&\eta_b&1\\0&\xi_c&\eta_c&1\end{array}\right],\qquad G=\left[\begin{array}{cccc}-1&0&0&0\\\xi_a^2+\eta_a^2&1&0&0\\\xi_b^2+\eta_b^2&0&1&0\\\xi_c^2+\eta_c^2&0&0&1\end{array}\right]$$

Matrix F acts on rows and kills quite all denominators, T acts on the last three columns and goes back to barycentrics while G acts on the first column to kill all square terms. After what everything simplifies nicely and leads to (10.3) □

Proposition 10.1.5. *The barycentric equation of circle with center $P=p:q:r$ and radius ω is :*

$$\left(a^2\,yz+b^2\,zx+c^2\,xy\right)+\omega^2\,\left(x+y+z\right)^2-\left(ux+vy+wz\right)\left(x+y+z\right)=0 \tag{10.4}$$

where quantities u,v,w are defined as :

$$\begin{array}{rll}u&\doteq|PA|^2=&\left(c^2q^2+b^2r^2+\left(b^2+c^2-a^2\right)qr\right)\div(p+q+r)^2\\v&\doteq|PB|^2=&\left(a^2r^2+c^2p^2+\left(c^2+a^2-b^2\right)rp\right)\div(p+q+r)^2\\w&\doteq|PC|^2=&\left(b^2p^2+a^2q^2+\left(b^2+a^2-c^2\right)pq\right)\div(p+q+r)^2\end{array} \tag{10.5}$$

Proof. Obvious from (10.1). The added value here is the emphasis on center and ω^2. It must be noticed that $u:v:w$ is not a point nor a line. Quantities u, v, w are strongly defined objects and are not defined up to a proportionality factor. They are to be considered exactly as ω^2, i.e. are of the same nature as a surface. It can be observed that u (or v or w) is zero-homogeneous wrt the barycentrics of point $P=p:q:r$. More details are given in Chapter 11 □

Proposition 10.1.6. ***Center****. The center of a circle defined by its equation (10.1) is given by :*

$$center=\frac{1}{2}vX(3)-\boxed{\mathcal{K}}\,.\,{}^tU=\left(\begin{array}{c}a^2\,S_a\\b^2\,S_b\\c^2\,S_c\end{array}\right)-\left(\begin{array}{ccc}a^2&-S_c&-S_b\\-S_c&b^2&-S_a\\-S_b&-S_a&c^2\end{array}\right)\cdot\left(\begin{array}{c}u\\v\\w\end{array}\right) \tag{10.6}$$

Proof. As stated in Definition 9.3.8, the center of a conic is the pole of the line at infinity wrt the conic. Computations are straightforward. Here, the standard choice for X(3), namely $vX\,(3)=a^2\left(b^2+c^2-a^2\right)::$ is not the efficient one when using the Al-Kashi matrix $\boxed{\mathcal{K}}$. It can be noticed that product $\boxed{\mathcal{K}}\,.\,{}^tU$ gives the orthodir of line U (i.e. the point at infinity in the orthogonal direction). Since the line of centers is orthogonal to the radical axis of Γ and Ω, this formula only says what coefficients to use. □

Proposition 10.1.7. ***Radius****. The radius of a circle defined by its equation (10.1) is given by :*

$$\omega^2=\frac{1}{16\,S^2}\left(U\cdot\boxed{\mathcal{K}}\cdot^t U-U.vX\,(3)+a^2b^2c^2\right) \tag{10.7}$$

Proof. Substract formula (10.1) from $|PX|^2$ obtained from (10.6) and Pythagoras formula. □

Remark 10.1.8. Everything in the center or the radius formulae should be used exactly as written, and not to a proportionality factor. A really projective formulation will be given later, with formula (11.12)

Definition 10.1.9. circlekit. Some usual square roots are given in Table 10.1, and some other notations in Table 10.2.

Example 10.1.10. Table 10.3 describes some of the usual circles in triangle geometry. For further information on many circles, refer to
`http://mathworld.wolfram.com/Circle.html`

name	#	value	where	
Lemoine	W_1	$\sqrt{a^2b^2+a^2c^2+b^2c^2}$	$e^{i\omega}=\dfrac{a^2+b^2+c^2+4\,iS}{2W_1}$	(10.11)
Brocard	W_2	$\sqrt{a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2}$	$\lvert OK\rvert=\dfrac{2\,W_2\,R}{a^2+b^2+c^2}$	(10.12)
Euler	W_3	$\sqrt{\sum_3 a^6-\sum_6 a^4b^2+3\,a^2b^2c^2}$	$\lvert OH\rvert=\dfrac{W_3}{4\,S}=\dfrac{R}{abc}\,W_3$	(10.13)
Furhmann	W_4	$\sqrt{\sum_3 a^3-\sum_6 a^2b+3\,abc}$	$\lvert HN\rvert=2\,W_4R\sqrt{\dfrac{1}{abc}}$	(10.14)

Table 10.1: Some usual square roots (circle kit 1)

#	value	name
s	$(a+b+c)\,/2$	half-perimeter
R	$\dfrac{abc}{\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}$	circumradius
S	$\dfrac{abc}{4R}$	area of triangle
ω	$\exp(i\,\omega)=\dfrac{a^2+b^2+c^2}{2W_1}+i\,\dfrac{2\,S}{W_1}$	Brocard angle
e	$\sqrt{\frac{a^4+b^4+c^4}{b^2c^2+c^2a^2+a^2b^2}-1}$	$\sqrt{1-4\,\sin^2\omega}$
r	$\dfrac{S}{s}=\dfrac{a\,b\,c}{2R\,(a+b+c)}$	inradius

Table 10.2: Some usual notations (circle kit 2)

Definition 10.1.11. Pencil of cycles. When $\mathcal{C}_1$and $\mathcal{C}_2$ are two circles, then $\lambda\mathcal{C}_1+\mu\mathcal{C}_2$ is also a cycle. The family generated from two given circles is called a pencil. Then all centers are on the same line, wich is orthogonal to the only line contained in the pencil (the so called radical axis). More details in Chapter 11

Definition 10.1.12. The **radical trace** of two non-concentric circles is the point of intersection of the radical axis of the circles and the line of the centers of the circles. (For examples, see X(I) for I = 6, 187, 1570, 2021-2025, 2030-2032.)

10.2 Antipodal Pairs on Circles

Proposition 10.2.1. Centers of similitude. *Here, all barycentrics are supposed to be in their normalized form. Let $(O_1,\,r_1)$, $(O_2,\,r_2)$ be two circles. Points U, V defined by :*

$$U=\frac{r_1O_2+r_2O_1}{r_1+r_2},\;V=\frac{r_1O_2-r_2O_1}{r_1-r_2}$$

are respectively the internal and the external centers of similitude of these two circles. When $X\in(O_1)$ then :

$$(r_1+r_2)\,U-r_2X\in(O_2)\quad\text{and}\quad(r_1-r_2)\,V+r_2X\in(O_2)$$

Proof. When $r_1=r_2$, point V defines a translation, not an homothecy. Otherwise, all steps are obvious. □

Proposition 10.2.2. *Suppose (O_1) and (O_2) are circles and that P, P' are antipodes on (O_1). Let $U=insim(O_1,\,O_2)$ and $V=exsim\,(O_1,\,O_2)$ be the respective internal and external center of*

Name	Center	Radius	
circumcircle	X(3)	R	10.4
incircle	X(1)	r	10.5
nine-point circle	X(5)	$\frac{1}{2}R$	10.6
conjugate circle	X(4)	$\sqrt{-S_a S_b S_c} \div 2S$	10.7
Longchamps circle	X(20)	$\sqrt{-S_a S_b S_c} \div S$	10.8
Bevan circle	X(40)	$2R$	10.9
Spieker circle	X(10)	$r/2$	10.10
Apollonius circle	X(970)	$\left(r^2+s^2\right) \div 4\,r$	10.11
1st Lemoine	X(182)	$R \div 2\cos\omega$	10.12
2nd Lemoine	X(6)	$abc/\left(a^2+b^2+c^2\right)$	10.13
Sin-triple-angle	X(49)	R_{sta}	10.14
Brocard circle	X(182)	$eR \div 2\cos\omega$	10.15
Brocard second	X(3)	eR	10.15
Orthocentroidal	X(381)	$\lvert OH\rvert/2$	10.16
Fuhrmann	X(355)	$\lvert HN\rvert/2$	10.17

Table 10.3: Some circles

similitude of circles (O_1) and (O_2). Define $Q = PU \cap P'V$ and $Q' = PV \cap P'U$. Then Q, Q' are antipodes on (O_2). Moreover, the lines PP' and QQ' are parallel.

Proof. The result is quite obvious, but giving a non-circular proof is not so obvious. To construct the centers of similitude, you can draw an arbitrary diameter $P_0O_1P_0'$ of the first circle and then the parallel diameter $Q_0O_2Q_0'$ of the second (arbitrary means : don't choose the center line !). Intersections $P_0Q_0 \cap P_0'Q_0'$ and $P_0'Q_0 \cap P_0Q_0'$ are the required points (in hand drawing, point V is better defined that way than using the centerline as in Coxeter (1989, page 70). Define a point $P \in (O_1)$ as "near" when it lies in the same half-plane wrt diameter P_0P_0' than O_2, and define it as "far" otherwise. Act conversely on (O_2). Then U-similitude converts far into far, near into near while V-similitude converts far into near and near into far.

Suppose that $P \in (O_1)$ is far. Then its V-image on (O_2) is the near point of $PV \cap (O_2)$ (call it R). The antipode P' is near, and its V image on (O_2) is the far point of $P'V \cap (O_2)$ (call it R'). By similitude, RR' is the diameter of (O_2) that is parallel to the diameter PP' of (O_1) and P, R' are far while P', R are near. The same thing can be done using the insimilitude. But now, the image of a far point is a far one: the U-image of P is the far point S of $PU \cap (O_2)$, while the U-image of P' is the near point S' of $P'U \cap (O_2)$. Here again, SS' is the diameter of Γ_2 that is parallel to the diameter PP' of (O_1) and we must have $S = R'$ and $S' = R$. □

In the following examples, suppose $P = p : q : r$ on the first circle.

10.3 Inversion in a circle

Definition 10.3.1. Two points X_1, X_2 are inverses in a given circle with center P and radius ω when P, X_1, X_2 are in straight line and $\left\langle \overrightarrow{PU_1} \mid \overrightarrow{PU_2} \right\rangle = \omega^2$.

Proposition 10.3.2. *The inverse of point $X = x : y : z$ in circle Ω having center $P = p : q : r$*

and radius ω is given by :

$$inv\left(X\right)=\left(\frac{\Gamma\left(X\right)}{\left(x+y+z\right)^{2}}+\frac{\Gamma\left(P\right)}{\left(p+q+r\right)^{2}}+\omega^{2}\right)\operatorname{nor}\left(P\right)$$
$$-\left(\frac{P\,.\,{}^{t}P}{\left(p+q+r\right)^{2}}\cdot\left(\begin{array}{ccc}0 & c^{2} & b^{2}\\ c^{2} & 0 & a^{2}\\ b^{2} & a^{2} & 0\end{array}\right)+\omega^{2}\right).\operatorname{nor}\left(X\right)\quad(10.8)$$

This formula is to be compared with formula (11.14) given in Theorem 11.4.13.

Proof. By definition, the inverse of point $X=u:v:w$ is given by :

$$\operatorname{nor}\left(inv\left(X\right)\right)=nor\left(P\right)+\left(nor\left(X\right)-nor\left(P\right)\right)\frac{\omega^{2}}{pytha\left(X,\,P\right)}$$

Collect terms according to their denominators in $pytha\left(X,\,P\right)\,nor\left(P\right)+\omega^{2}\,\left(nor\left(X\right)-nor\left(P\right)\right)$. Then most terms of (10.8) become apparent. The matricial one appears using

$$\left({}^{t}P\,.\,X\right)P=\left(P\,.\,{}^{t}P\right).\,X$$

□

Remark 10.3.3. When circle Ω is given by its equation (10.1) and P, ω are obtained from (10.6) and (10.7), the following identity can be useful :

$$\frac{\Gamma\left(P\right)}{\left(p+q+r\right)^{2}}+\omega^{2}=\frac{2\,a^{2}b^{2}c^{2}-U\,.\,vX\left(3\right)}{16\,S^{2}}$$

(remember: U and $vX\left(3\right)$ are "as is" and not "up to a proportionality factor").

10.4 Circumcircle

Definition 10.4.1. The circumcircle is the circle through A,B,C. Perspector is X_6 and center X_3. Equation is :

$$a^{2}yz+b^{2}zx+c^{2}xy=0$$

Its standard parameterization is (5.15), i.e. :

$$\frac{a^{2}}{\sigma-\tau}:\frac{b^{2}}{\tau-\rho}:\frac{c^{2}}{\rho-\sigma}\mid\rho:\sigma:\tau\neq\mathcal{L}_{\infty}$$

Lemma 10.4.2. *The distance $\left|PX_3\right|$ to center from any point is given by :*

$$\left|PX_{3}\right|^{2}=R^{2}-\frac{a^{2}qr+b^{2}rp+c^{2}pq}{(p+q+r)^{2}}$$

Proof. Direct inspection using Theorem 5.2.4. As it should be, $\left|X_3X_3\right|=0$ while the equation of the circumcircle is $\left|PX_3\right|^2=R^2$. □

Proposition 10.4.3. *For any finite point, other than the circumcenter X_3, the inverse-in-circumcircle of $P=p:q:r$ has barycentrics $u:v:w$ obtained cyclically from :*

$$u=-p^{2}+\frac{c^{2}-a^{2}}{b^{2}}pq+\frac{b^{2}-a^{2}}{c^{2}}pr+\frac{a^{2}\left(b^{2}+c^{2}-a^{2}\right)}{b^{2}c^{2}}qr$$

1	36	24	403	54	1157	352	353	1692	3053
2	23	25	468	55	1155	371	2459	2482	2930
4	186	26	2072	56	1319	372	2460	2935	3184
5	2070	27	2073	57	2078	399	1511	3110	3286
6	187	28	2074	58	1326	667	1083	3438	3480
10	1324	29	2075	67	3455	859	3109	3439	3479
15	16	32	1691	115	2079	1054	1283	3513	3514
20	2071	35	484	131	2931	1145	2932		
21	1325	39	2076	182	2080	1384	2030		
22	858	40	2077	237	1316	1687	1688		

Proof. Points X(3), P, U are on the same line, and distance from X(3) to U is $R^2/|PX_3|$. Therefore, in normalized barycentrics, we have : $u = x_3 + (p - x_3)\, R^2/|PX_3|^2$. □

Remark 10.4.4. On ETC $n \leq 3587$, there are :

- 258 named points that belongs to Γ
- 47 pairs of "true" inverses that both are named
- 220 named points of Γ that have a named isogonal conjugate (among the 229 points of $\mathcal{L}_\infty$)
- 62 pairs of named antipodal points

10.5 Incircle

Definition 10.5.1. The incircle is one of the four circles that are tangent to the sidelines. This circle is inside the triangle, and also inside the nine point circle (these circles are tangent). Center X(1), perspector X(7), radius $r = S/s = abc \div 2R\,(a+b+c)$, equation :

$$\Gamma = \mathcal{L}_\infty \times \frac{1}{4} \sum (b-a+c)^2\, x \tag{10.9}$$

Proof. Well known properties. □

Remark 10.5.2. On ETC $n \leq 3587$, there are :

- 39 named points on the incircle
- 7 pairs of "true" inverses that both are named
- 10 pairs of named antipodal points

Proposition 10.5.3. *Centers of similitude with the circumcircle are in=X(55), and ex=X(56). When $p:q:r \in \Gamma$, then Q, Q' are antipodal points on the incircle :*

$$Q = \left((b-c)^2\, p + a^2 q + a^2 r\right)(b+c-a)\,,\ \text{etc}$$

$$Q' = \frac{(b+c)^2\, p + a^2 q + a^2 r}{b+c-a},\ \text{etc}$$

Proof. Proposition 10.2.2. □

Proposition 10.5.4. ***Incircle transform.*** *Let $U = u:v:w$ be a point other than the symmedian point, X_6. Then reflection of $\mathcal{C}_7$ (the intouch triangle) in the line UX_1 is perspective with triangle ABC. The isogonal conjugate of the corresponding perspector is called the incircle transform of U. Its barycentrics are :*

$$IT\,(u:v:w) = \frac{a^2\,(bw-cv)^2}{b+c-a} : \frac{b^2\,(cu-aw)^2}{a+c-b} : \frac{c^2\,(av-bu)^2}{b+a-c}$$

and this point is on the incircle.

Proof. Line UX_1 is a diameter of the incircle and the reflected triangle $\mathcal{T}$ is also inscribed in the incircle. The barycentrics of UX_1 are $[bw - cv, cu - aw, av - ub]$. Reflection in this line is obtained using (5.23), and barycentrics of $\mathcal{T}$ are obtained. Perspectivity and perspector are easily computed and conclusion follows by substituting in the incircle equation. □

Remark 10.5.5. In ETC another formula is given... and the point is also on the incircle :

$$IT2\left(U\right) = \frac{a^2\left(b^2 w - c^2 v\right)^2}{b^2c^2\left(b+c-a\right)} : \frac{b^2\left(c^2 u - a^2 w\right)^2}{a^2c^2\left(c+a-b\right)} : \frac{c^2\left(a^2 v - b^2 u\right)^2}{a^2b^2\left(a+b-c\right)}$$

One has $IT\left(X\right) = IT\left(U\right)$ when U, X aligned with X_1 while $IT2\left(X\right) = IT\left(U\right)$ when U, X aligned with X_6.

10.6 Nine-points circle

Definition 10.6.1. The nine point circle is the circumcircle of the orthic triangle. It goes also through the six midpoints of the orthocentric quadrangle $ABCH$. Center X(5), radius $R/2$ (half the ABC circumradius), perspector :

$$\frac{1}{a^4 - a^2b^2 - a^2c^2 - b^2c^2} : \frac{1}{b^4 - a^2c^2 - b^2c^2 - a^2b^2} : \frac{1}{c^4 - a^2c^2 - b^2c^2 - a^2b^2}$$

is not named, equation :

$$\Gamma = \mathcal{L}_\infty \times \frac{1}{2}\left(xS_a + yS_b + zS_c\right) \tag{10.10}$$

Proposition 10.6.2. *Centers of similitude with Γ are in=X(2), and ex=X(4). Ω intersects Γ when ABC is not acute. Radical trace X(468), direction of center axis (Euler line) X(30), direction of radical axis X(523). Standard parameterization (similitude from circumcircle) :*

$$U \simeq \begin{pmatrix} \left(\sigma-\tau\right)\left(c^2\tau - b^2\sigma + b^2\rho - c^2\rho\right) \\ \left(\tau-\rho\right)\left(a^2\rho - c^2\tau + c^2\sigma - a^2\sigma\right) \\ \left(\rho-\sigma\right)\left(b^2\sigma - a^2\rho + a^2\tau - b^2\tau\right) \end{pmatrix}$$

Remark 10.6.3. On ETC $n \leq 3587$, there are :

- 37 named points on the nine points circle
- 9 pairs of "true" inverses that both are named
- 12 pairs of named antipodal points

Proposition 10.6.4 (Feuerbach)**.** *The nine-point circle is tangent to the incircle and the three excircles.*

Proof. Use (10.9) and (10.10) to obtain the radical axis. □

10.7 Conjugate circle

Definition 10.7.1. The conjugate circle is the only circle whose matrix is diagonal (and triangle ABC is autopolar). Center X(4), the orthocenter, radius $\sqrt{-S_aS_bS_c}/2S$, equation :

$$\sum S_a x^2 = \mathcal{L}_\infty \times \left(x\,S_a + y\,S_b + z\,S_c\right) - \Gamma = 0$$

Proposition 10.7.2. *This circle belongs to the same pencil as the circum and the nine-points circles. This circle is real only when triangle ABC is not acute. Therefore, no named points can belong to this circle.*

Proposition 10.7.3. *The conjugate circle is the locus of the centers of the inscribed rectangular hyperbolas (cf Section 9.5).*

10.8 Longchamps circle

Definition 10.8.1. The Longchamps circle of ABC is the conjugate circle of the antimedial triangle.

Proposition 10.8.2. *The Longchamps circle is the locus of the auxiliary points of the inscribed rectangular hyperbolas (cf Section 9.5).*

10.9 Bevan circle

Definition 10.9.1. The Bevan circle is the circumcircle of the excentral triangle. Perspector X(57), center X(40), radius $2R$, equation :

$$\Gamma = \mathcal{L}_\infty \times (-bcx - acy - abz)$$

Proposition 10.9.2. *Centers of similitude with Γ are in=X(165), and ex=X(1). Radical trace X(1155), center axis X(517), radical axis X(513). Moses parameterization leads to Q (bad looking) and $Q' = -(a+2\,b+2\,c)\,p + aq + ra$.*

Remark 10.9.3. On ETC $n \leq 3587$, there are :

- 9 named points on the Bevan circle, namely : 1054, 1282, 1768, 2100, 2101, 2448, 2449, 2948, 3464
- 5 pairs of "true" inverses that both are named
- 2 pairs of named antipodal points

10.10 Spieker circle

Definition 10.10.1. Spieker circle is the incircle of the medial triangle. Perspector X(2), center X(10), radius $r/2$ (half the ABC inradius), equation :

$$\Gamma = \mathcal{L}_\infty \times \frac{1}{16}\sum \left(5\,c^2 + 5\,b^2 - 3\,a^2 + 2\,ac + 2\,ab - 6\,bc\right) x$$

Proposition 10.10.2. *Centers of similitude with Γ are in=X(958), and ex=X(1376). Radical trace not named, center axis X(515), radical axis X(522). Moses parameterization leads to :*

$$Q = (b+c-a)\left(\left(ab^2 + ac^2 + b^3 - b^2c - c^2b + c^3\right)p + a\,(q+r)\left(a^2 + ab + ac + 2\,bc\right)\right)$$

$$Q' = \left(ab^2 + c^2a + b^2c + c^2b - c^3 - b^3\right)p + a\,(q+r)\left(+a^2 - ab - ac + 2\,bc\right), \text{ etc}$$

Centers of similitude with the incircle are in=X(8) and ex=X(2).

Remark 10.10.3. On ETC $n \leq 3587$, there are :

- 8 named points on the Spieker circle, namely : 3035, 3036, 3037, 3038, 3039, 3040, 3041, 3042
- no pairs of "true" inverses that both are named
- 2 pairs of named antipodal points [3035,3036], [3042,3042]

10.11 Apollonius circle

Definition 10.11.1. The **Apollonius circle** is tangent to the three excircles and encloses them. Center X(970), radius $(abc + \sum_6 a^2b) \div 8S$, perspector not named, equation :

$$\Gamma = \mathcal{L}_\infty \times \frac{a+b+c}{4}\sum \frac{a^2 + ab + ac + 2\,bc}{a}\,x$$

Proposition 10.11.2. *Centers of similitude with Γ are in=X(573), and ex=X(386). Radical trace not named, center axis X(511), radical axis X(512). Moses parameterization leads to bad looking Q and*

$$Q' = -\left(b+c\right)^{2}\left(a+c\right)\left(a+b\right)p + \left(ab+ac+bc+b^{2}+c^{2}\right)a^{2}q + r\left(ab+ac+bc+b^{2}+c^{2}\right)a^{2}, \text{ etc}$$

Centers of similitude with the nine-points circle are in=X(10) and ex=X(2051).

Remark 10.11.3. On ETC $n \leq 3587$, there are :

- 8 named points on the Apollonius circle, namely : 2037, 2038, 3029, 3030, 3031, 3032, 3033, 3034
- no pairs of "true" inverses that both are named
- 1 pairs of named antipodal points [2037, 2038].

10.12 First Lemoine circle

Definition 10.12.1. The **first Lemoine circle** of ABC is obtained as follows. Draw parallels to the sidelines of ABC through Lemoine point X_6. The six intersections of these lines with sidelines are concyclic on the required circle. The following surd is useful :

$$W_1 = \sqrt{a^2b^2 + a^2c^2 + b^2c^2} \tag{10.11}$$

Proposition 10.12.2. *Center is X(182) (i.e. OK midpoint), radius $R \div 2\cos\omega = W_1\, R/\left(a^2+b^2+c^2\right)$, perspector :*

$$\frac{a^2}{2\,a^2b^2+2\,a^2c^2+b^2c^2}, \frac{b^2}{a^2c^2+2\,a^2b^2+2\,b^2c^2}, \frac{c^2}{a^2b^2+2\,a^2c^2+2\,b^2c^2}$$

is not named, equation :

$$\Gamma = \mathcal{L}_\infty \times \frac{1}{\left(a^2+b^2+c^2\right)^2} \sum \left(b^2+c^2\right) b^2c^2x$$

This circle is concentric with and external to the first Brocard circle.

Proof. Difference of squared radiuses factors into $\left(abc/\left(a^2+b^2+c^2\right)\right)^2$. □

Proposition 10.12.3. *Centers of similitude with Γ are in=X(1342), and ex=X(1343). Radical trace X(1691), center axis X(511), radical axis X(512). Moses parameterization leads to bad looking Q and Q'. Poncelet centers of the pencil : X(1687) (inside) and X(1688) outside.*

Proof. In order to see that X(1687) is inside, compute $\Omega\left(X(182)\right) \times \Omega\left(X\left(1687\right)\right)$ and obtain a quantity that is clearly positive. □

Remark 10.12.4. On ETC $n \leq 3587$, there are :

- 2 named points on the Apollonius circle, namely : 1662, 1663 (intersection with the Brocard axis, X(3)X(6)).
- 7 pairs of "true" inverses that both are named :

$$\left[\begin{array}{ccccccc} 3 & 6 & 32 & 39 & 371 & 372 & 1687 \\ 2456 & 1691 & 1692 & 2458 & 2461 & 2462 & 1688 \end{array}\right]$$

- 1 pairs of named antipodal points [1662, 1663].

10.13 Second Lemoine circle

Definition 10.13.1. The **second Lemoine circle** of ABC is obtained as follows. Draw parallels to the sidelines of orthic triangle through Lemoine point X_6. The six intersections of these lines with sidelines are cocyclic on the required circle.

Proposition 10.13.2. *Center is X(6) itself, radius* $abc/\left(a^2+b^2+c^2\right)$*, perspector X(3527), equation :*

$$\Gamma = \mathcal{L}_\infty \times \frac{2}{\left(a^2+b^2+c^2\right)^2}\sum b^2c^2\left(b^2+c^2-a^2\right)x$$

Centers of similitude with Γ *are in=X(371), and ex=X(372). Radical trace X(1692), center axis X(511), radical axis X(512). Moses parameterization leads to bad looking* Q *and* Q'*. Poncelet centers of the pencil are involving radical* $\sqrt{6\sum a^2b^2-5\sum a^4}$ *and are not named. Moreover, the second Lemoine circle is bitangent to the Brocard ellipse.*

Remark 10.13.3. On ETC $n \leq 3587$, there are :

- 2 named points on the second Lemoine circle, namely : 1666, 1667 (intersection with the Brocard axis, X(3)X(6)).
- 5 pairs of "true" inverses that both are named :

$$\left[\begin{array}{ccccc} 3 & 576 & 1316 & 1351 & 2452 \\ 1570 & 1691 & 2451 & 1692 & 3049 \end{array}\right]$$

- 1 pairs of named antipodal points [1666, 1667].

10.14 Sine-triple-angle circle

Definition 10.14.1. Define inscribed triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ by the properties :

$$\angle(AB,\, AC) = \angle(B_1A,\, B_1C_1) = \angle(C_2B_2,\, C_2A)\,,\ \text{etc}$$

the idea being to obtain isosceles "remainders". Then all the six vertices are on the same circle.

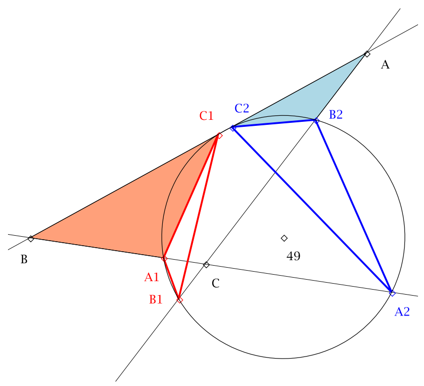


Figure 10.1: Sin triple angle circle

Proof. Using the tangent formula, the six points are easily obtained. $\mathcal{T}_1$ is a central triangle, and each vertex of $\mathcal{T}_2$ is obtained by a transposition.

$$A_1 \simeq \left(\begin{array}{c} 0 \\ \left(a^2-ac-b^2\right)\left(a^2+ac-b^2\right)\left(a^2+b^2-c^2\right) \\ \left(a^2-bc-c^2\right)\left(a^2+bc-c^2\right)c^2 \end{array}\right) \qquad \square$$

Proposition 10.14.2. *Center X(49), perspector not named, equation horrific, radius* $R_{tsa} = R^3 / \left(|OH|^2 - 2\,R^2 \right)$*. Centers of similitude with* Γ *are in=X(1147), and ex=X(184), direction of radical axis X(924). Moses parameterization leads to bad looking* Q *and :*

$$Q' = a^2 \left(a^2 - c^2\right) \left(a^2 - b^2\right) p - a^4 \left(b^2 + c^2 - a^2\right) (q + r)\,, \text{ etc}$$

Remark 10.14.3. On ETC $n \leq 3587$, there are :

- 6 named points on the Sine Triple Angle circle, namely : 3043, 3044, 3045, 3046, 3047, 3048
- 0 pairs of "true" inverses that are both named
- 1 pairs of named antipodal points [3043, 3047].

10.15 Brocard 3-6 circle and second Brocard circle

Definition 10.15.1. Brocard circle has $[X_3, X_6]$ for diameter. Center X(182). Radius $e\,R \div 2\cos\omega = W_2\,R/\left(a^2 + b^2 + c^2\right)$ where :

$$W_2 = \sqrt{a^4 + b^4 + c^4 - (b^2c^2 + a^2b^2 + a^2c^2)} \tag{10.12}$$

while the perspector :

$$\frac{a^2}{2a^4 + b^2c^2} : \frac{b^2}{2b^4 + a^2c^2} : \frac{c^2}{2c^4 + a^2b^2}$$

is not named. Equation :

$$\Gamma = \mathcal{L}_\infty \times \frac{1}{(a^2 + b^2 + c^2)} \left(b^2c^2x + a^2c^2y + a^2b^2z\right)$$

Proposition 10.15.2. *Centers of similitude with* Γ *are in=X(1340), and ex=X(1341). Radical trace X(187), center axis X(511), radical axis X(512). Moses parameterization leads to :*

$$Q = \begin{pmatrix} a^2 \left(a^4 - a^2c^2 - a^2b^2 - 2\,b^2c^2\right)(p + q + r) \\ \pm W\left(\left(a^2b^2 + a^2c^2 + 2\,b^2c^2 - b^4 - c^4\right)p - a^2\left(b^2 + c^2 - a^2\right)(q + r)\right) \end{pmatrix}, \text{ etc}$$

Poncelet centers of the pencil are X(15) and X(16), the isodynamic points.

Remark 10.15.3. On ETC $n \leq 3587$, there are :

- 4 named points on the first Brocard circle, namely : 3, 6, 1083, 1316. Moreover, this circle goes through the Brocard points (cf 5.7.1).
- 47 pairs of "true" inverses that are both named
- 1 pairs of named antipodal points [3, 6].

Definition 10.15.4. Second Brocard circle is centered on X(3) and goes through the Brocard's centers. Radius $eR = W_2\,R/\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$, while the perspector :

$$\frac{a^2}{2a^4 - a^2b^2 - a^2c^2 + b^2c^2}, \text{ etc}$$

is not named. Equation :

$$\Gamma = (x + y + z)^2 \, \frac{a^2b^2c^2}{a^2b^2 + a^2c^2 + b^2c^2}$$

Remark 10.15.5. On ETC $n \leq 3587$, there are :

- 6 named points on the second Brocard circle, namely : 1670, 1671, 2554, 2555, 2556, 2557. Moreover, this circle goes through the Brocard points (cf 5.7.1).

- 17 pairs of "true" inverses that are both named

6	39	62	3106	1340	3558	2026	2561
15	3105	76	99	1341	3557	2027	2560
16	3104	182	3095	1666	2563		
32	3094	371	3103	1667	2562		
61	3107	372	3102	1689	1690		

- 3 pairs of named antipodal points [1670, 1671], [2554, 2555], [2556, 2557].

10.16 Orthocentroidal 2-4 circle

Definition 10.16.1. Orthocentroidal circle has $[X_2,\, X_4]$ for diameter. Center is X(381) and radius $R\,W_3 \div 3abc$ where :

$$W_3 = \sqrt{a^6+b^6+c^6-\left(a^4b^2+a^4c^2+a^2b^4+a^2c^4+b^4c^2+b^2c^4\right)+3a^2b^2c^2} \tag{10.13}$$

while the perspector :

$$\frac{1}{b^2c^2+2\,a^2\left(b^2+c^2-a^2\right)}$$

is not named. Equation :

$$\Gamma = \mathcal{L}_\infty \times \frac{1}{3}\,\left(\left(b^2+c^2-a^2\right)x+\left(c^2+a^2-b^2\right)y+\left(a^2+b^2-c^2\right)z\right)$$

Proposition 10.16.2. *Centers of similitude with Γ are in=X(1344), and ex=X(1345). Radical trace X(468), center axis X(30), radical axis X(523). Moses parameterization leads to :*

$$Q = \left(\begin{array}{c} abc\left(a^4+a^2b^2+a^2c^2-2\,b^4+4\,b^2c^2-2\,c^4\right)(p+q+r) \\ \pm W\left(\left(a^2b^2+a^2c^2+2\,b^2c^2-b^4-c^4\right)p-a^2\left(b^2+c^2-a^2\right)(q+r)\right)\end{array}\right),\ \text{etc}$$

Poncelet points are real when triangle is acute.

Remark 10.16.3. On ETC $n \leq 3587$, there are :

- 2 named points on the orthocentroidal circle, namely (antipodal) 2, 4
- 45 pairs of "true" inverses that are both named

Proposition 10.16.4. *Let $A_HB_HC_H$ be the feet of the altitudes, and* $A' = (A+2A_H)/3$, etc. *This circle goes through points A', B', C'.*

10.17 Fuhrmann 4-8 circle

Definition 10.17.1. Fuhrmann circle has $[X_4,\, X_8]$ for diameter (see also). Center X(355), radius $R\,W_4 \div \sqrt{abc}$ where :

$$W_4 = \sqrt{a^3+b^3+c^3-\left(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2\right)+3\,abc} \tag{10.14}$$

perspector not named (and not handy), equation :

$$\Gamma = \mathcal{L}_\infty \times \frac{1}{a+b+c}\left(a\left(b^2+c^2-a^2\right)x+b\left(c^2+a^2-b^2\right)y+c\left(a^2+b^2-c^2\right)z\right)$$

Remark 10.17.2. The only named points of this circle are X(2) and X(4). Five inverse pairs are :

1	11	72	2475	3434
80	1837	3419	3448	3436

10.18 Kiepert RH and isosceles adjunctions

Proposition 10.18.1. ***Kiepert RH****. Chose angle ϕ and construct isosceles triangles $BA'C$, $CB'A$, $AC'B$ with basis angle $\angle(BC, BA') = \phi$ ($\phi < 0$ when A' is outside). Then triangles ABC and $A'B'C'$ are perspective wrt a point $N(\phi)$:*

$$N(\phi) \simeq \frac{1}{2S\,\cot\phi - S_a},\ \text{etc} \simeq \frac{a}{\sin(A-\phi)},\ \text{etc} \tag{10.15}$$

and Kiepert RH is the locus of such points. Perspector of this conic is X(523), DeLongchamp point at infinity, and center is X(115).

Proof. One has : $A' = a^2\tan\phi : 2\,S - S_c\tan\phi : 2\,S - S_b\tan\phi$. □

Proposition 10.18.2. *For a given K, the points A'B'C' are on the same cubic as the vertices, the inexcenters, orthocenter, circumcenter and the points $A'B'C'$ relative to the opposite value of K. This cubic can be written as $\left(K^2+1\right)K001 + \left(3-K^2\right)K003$ where $K001$ and $K003$ are the standardized equations of, respectively, the Neuberg and the McKay cubics, as given in Proposition 15.3.11 and Proposition 15.3.10. The pole is X(6), while pivot is $3\,vX(2) - K^2\,vX(20)$ i.e. : $\left(1+K^2\right)s_1\,s_3 : \left(3-K^2\right)s_3 : \left(1+K^2\right)s_2$.*

Proof. Details are given in Proposition 15.3.8. □

Remark 10.18.3. Points at infinity of Kiepert RH are parameterized by :

$$\begin{aligned}\cot\phi &= \frac{-1}{3}\left(1+\frac{2\,W_2}{a^2+b^2+c^2}\right)\cot(\omega) = \frac{-1}{12S}\left(a^2+b^2+c^2+2\,W_2\right)\\ W_2 &= \sqrt{a^4-a^2b^2-a^2c^2+b^4-b^2c^2+c^4} \qquad \text{is the Brocard radical}\end{aligned}$$

Example 10.18.4. Fixed values of angle ϕ can be obtained by adding some regular shape to each side of the reference triangle. In Figure 10.2, a "cake server" (Pelle à Tarte) like $AFUGC$, made of a square $AFGC$ and an equilateral triangle FUG defines angle $\phi = \left(\overbrace{AU,\ AC}\right) = -75°$. When the triangle is inside the square, we obtain a Pelle Pliée (folded shovel) like $AFWGC$ that defines angle $\phi = \left(\overbrace{AU,\ AC}\right) = -15°$.

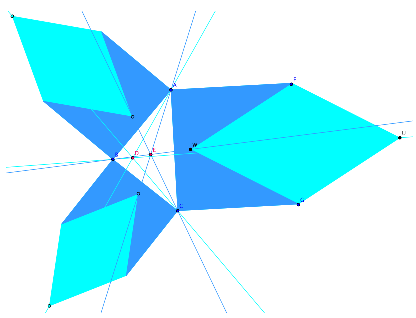


Figure 10.2: Cake server (Pelle a Tarte)

Example 10.18.5. Arbelos. Another idea to obtain some adding object is as follows. Divide $[AB]$ in a given ratio, obtaining D. Use the same ratio to obtain E, F. Draw the half circles and perpendicular DM, then the tangent circles. The triangle of the centers is perspective with ABC if and only if the ratio is $\left(\sqrt{5}-1\right)/2$. Perspector is X(2672). And belongs to Kiepert RH. Therefore, we have isosceles triangle. Value is $\pm\tan\phi = 3-\sqrt{5}$ (plus is inward, minus is outward).

Example 10.18.6. When using the center of a regular triangle, square, pentagon, we have $\phi = 30°$, $\phi = 45°$, $\phi = 54°$; when using the farthest vertex (or the midpoint of the farthest side), $\phi = 60°$, $\phi = at2 \doteq \arctan 2$, $\phi = 72°$. We even have points relative to $at3 = \arctan 3$. Points are collected into 2×2 blocks corresponding to $-\phi$, $\phi - 90°$, $90° - \phi$, ϕ.

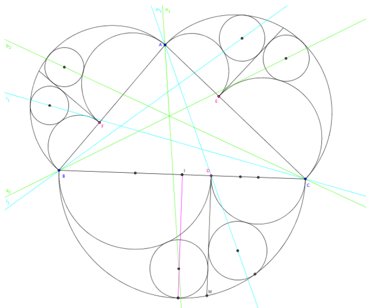


Figure 10.3: Arbelos configuration

$-90°$	0	$-75°$	$-15°$	$-72°$	$-18°$	$-at3$		$-67.5°$	$-22.5°$
4	2	3391	3366	1139	3370	1327	?	3387	3373
4	2	3367	3392	1140	3397	1328	?	3374	3388
$90°$	0	$75°$	$15°$	$72°$	$18°$	$at3$		$67.5°$	$22.5°$
H	G	PaT	PPl	$Penta$					

$-at2$	$at2-90°$	$-60°$	$-30°$	$-54°$	$-36°$		$-arb$	$-45°$	$-45°$
1131	3316	13	17	3381	?	?	2671	485	485
1132	3317	14	18	?	3382	?	2672	486	486
$at2$	$90°-at2$	$60°$	$30°$	$54°$	$36°$		arb	$45°$	$45°$
		$Fermat$				$Arbelos$		$Vecten$	

Example 10.18.7. Brocard angle is often involved since :

$$P_\phi \simeq \frac{1}{(b^2+a^2+c^2)\cot(\phi)-S_a\cot(\omega)}, \text{ etc}$$

This leads to four new blocks :

$-2\,\omega$	$2\,\omega-\frac{\pi}{2}$	$-\omega$	$\omega-\frac{\pi}{2}$	$-\frac{1}{2}\,\omega$	$\frac{1}{2}\,\omega-\frac{\pi}{2}$	$-\frac{\pi}{4}-\frac{1}{2}\,\omega$	$\frac{1}{2}\,\omega-\frac{\pi}{4}$
3407	3399	83	262	1676	?	?	2009
1916	3406	76	98	?	1677	2010	?
$2\,\omega$	$\frac{\pi}{2}-2\,\omega$	ω	$\frac{\pi}{2}-\omega$	$\frac{1}{2}\,\omega$	$\frac{\pi}{2}-\frac{1}{2}\,\omega$	$\frac{\pi}{4}+\frac{1}{2}\,\omega$	$\frac{\pi}{4}-\frac{1}{2}\,\omega$
$Gibert$		$Brocard$		$Lemoine\ circle$		$Galatly\ circle$	

Proposition 10.18.8. *Let σ be a constant value. Then all lines $N(\phi)\,N(\sigma-\phi)$ are passing through point $T(\sigma)$ where :*

$$T(\sigma) \simeq \left(\begin{array}{c} a^2S_a \\ b^2S_b \\ c^2S_c \end{array}\right) + 2S\,\cot\sigma \left(\begin{array}{c} a^2 \\ b^2 \\ c^2 \end{array}\right)$$

Therefore all the $N(\phi)\,N(-\phi)$ lines (rows in a block) are passing through X(6), the Lemoine point. And all the $N(\phi)\,N\left(\frac{\pi}{2}-\phi\right)$ lines (columns in a block) are passing through X(3) the circumcenter. All the $T(\sigma)$ are on the Brocard axis X(3)X(6).

Proof. Obvious from (10.15). □

Proposition 10.18.9. *Let δ be a constant value. Then all lines $N(\phi)\,N(\phi+\delta)$ are tangent to a conic $D(\delta)$. When $\delta=0$ this conic is the Kiepert RH itself. When $\delta=\pi/2$ (diagonals in a block), the conic reduces to a real point : the Euler center X(5). In the general case, $D(\delta)=16S^2\left(\prod\left(a^2-b^2\right)\right)\cot^2(\delta)\ D(0)+D(\pi/2)$.*

Proof. This can be proved using the usual techniques : differentiate and wedge. When δ is rational wrt π, conics $D(\delta)$ and $D(0)$ are in a Steiner configuration. □

Lemma 10.18.10. *Consider circle $\mathcal{C}_0 : x^2 + y^2 = 1$ and points $U = (u, 0)$, $V = (v, 0)$ in the Cartesian plane. When $u + v \neq 0$, these points are the centers of similitude of circles $\mathcal{C}_0$ and $\mathcal{C}(P, \rho)$ where*

$$P = \left(\frac{2\,uv}{u+v}, 0\right), \ \rho = \left|\frac{u-v}{u+v}\right|$$

Proof. Consider $M = (0, +1)$ and $N = (0, -1)$. Their counterparts in circle $\mathcal{C}$ are obtained by the intersections $J = UM \cap VN$ and $K = UN \cap VM$. We have $P = (J + K)/2$ and $\rho = |J - K|/2$. □

Proposition 10.18.11. *A circle $\mathcal{C}(P, \rho)$ can be found such that points $U = N(\phi)$, $V = N(\phi + \pi/2)$ are the centers of similitude between $\mathcal{C}$ and the nine-points circle. We have :*

$$\begin{aligned} P &\simeq \cot(2\,\phi)\begin{pmatrix} a^2 S_a \\ b^2 S_b \\ c^2 S_c \end{pmatrix} - 2\,S \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \cong \cot(2\,\phi)\ X_3 - 2\,S\,X_6 \\ \rho &= \frac{abc\,\sqrt{1+\cot^2(2\,\phi)}}{a^2+b^2+c^2-4\,S\cot(2\,\phi)} \end{aligned}$$

Proof. These two points are aligned with E=X(5). Define $W = (U + V)/2$. From the above lemma, we have :

$$P = E + k\,(U - E)\ ;\ \rho = (1-k)\,\frac{R}{2} \qquad \text{where } k = \frac{V-E}{W-E}$$

Computations are greatly simplified when remarking that the results depends not really from $\cot\phi$ itself, but rather from $\cot\phi - 1/\cot\phi$. That is the reason why all these formulae are involving $\cot(2\phi)$. □

10.19 Cyclocevian conjugate

Definition 10.19.1. Two points are called cyclocevian conjugates when their cevian triangles share the same circumcircle. This definition has to be compared with cyclopedal conjugacy, see Section 6.3.

Proposition 10.19.2 (Terquem)**.** *Each point not on the sidelines has a cyclocevian conjugate. Its barycentrics are given by (Grinberg, 2003a) :*

$$U = cyclocevian\,(P) = (isot \circ anticomplement \circ isog \circ complement \circ isot)\,(P)$$

Proof. This proposition asserts that the other intersections of the circumcircle of $A_P B_P C_P$ with the sidelines are the vertices of another cevian triangle. As it should be, the formula is involutory. The center of the common circumcircle is the middle of $[PU]$. □

Example 10.19.3. Some examples :

point	code	bary	cycc	circumcenter
Gergonne	$X(7))$	$1/(-a+b+c)$	$X(7)$	$X(1)$
centroid	$X(2)$	1	$X(4)$	$X(5)$
orthocenter	$X(4)$	$1/\left(-a^2+b^2+c^2\right)$	$X(2)$	$X(5)$
Nagel	$X(8)$	$-a+b+c$	$X(189)$	$X(1158)$

Point X_7 is the only center that is invariant by *cyclocevian*. Three other points share this property, obtained by changing one of the a, b, c into its opposite in the barycentrics of X_7.

ϕ	U	V	P	ρ	*name*
0	2	4	3	R	*circum*
$\pi/12$	3392	3391	16	$\dfrac{2\sqrt{3}abc}{12\,S-\sqrt{3}\,(a^2+b^2+c^2)}$	
$-\pi/12$	3366	3367	15	$\dfrac{2\sqrt{3}abc}{12\,S+\sqrt{3}\,(a^2+b^2+c^2)}$	
$\pi/10$	3397	1139	3393		
$-\pi/10$	3370	1140	3379		
$\arctan(3)$	1328	?	?	$\dfrac{5\,abc}{16\,S+3\,(a^2+b^2+c^2)}$	
$-\arctan(3)$	1327	?	?	$\dfrac{5\,abc}{16\,S-3\,(a^2+b^2+c^2)}$	
$\pi/8$	3388	3387	372	$\dfrac{\sqrt{2}abc}{4\,S-(a^2+b^2+c^2)}$	
$-\pi/8$	3373	3374	371	$\dfrac{\sqrt{2}abc}{4\,S+(a^2+b^2+c^2)}$	
$\arctan(2)$	1132	3316	?	$\dfrac{5\,abc}{12\,S+4\,(a^2+b^2+c^2)}$	
$-\arctan(2)$	1131	3317	?	$\dfrac{5\,abc}{12\,S-4\,(a^2+b^2+c^2)}$	
$\pi/6$	18	13	62	$\dfrac{2\,abc}{4\,S-\sqrt{3}\,(a^2+b^2+c^2)}$	
$-\pi/6$	17	14	61	$\dfrac{2\,abc}{4\,S+\sqrt{3}\,(a^2+b^2+c^2)}$	
$\pi/5$	3382	3381	3395	$--$	
$-\pi/5$	?	?	3368	$--$	
$\arctan\left(3-\sqrt{5}\right)$	2672	?	?	$\dfrac{3\left(4\sqrt{5}+5\right)abc}{44\,S-2\left(9+5\sqrt{5}\right)(a^2+b^2+c^2)}$	
$-\arctan\left(3-\sqrt{5}\right)$	2671	?	?	$\dfrac{3\left(4\sqrt{5}+5\right)abc}{44\,S+2\left(9+5\sqrt{5}\right)(a^2+b^2+c^2)}$	
$\pi/4$	486	485	6	$\dfrac{abc}{a^2+b^2+c^2}$	2° *Lemoine*
$\operatorname{arccot}\dfrac{(a+b+c)^2}{4\,S}$	10	2051	970	$\dfrac{r^2+p^2}{4\,r}$	*Apollonius*
$-\operatorname{arccot}\dfrac{(a+b+c)^2}{4\,S}$	?	?	?		
$2\,\omega$	1916	3399	?	$\dfrac{\left(a^2b^2+a^2c^2+b^2c^2\right)R}{a^4+b^4+c^4+a^2c^2+a^2b^2+b^2c^2}$	
$-2\,\omega$	3407	3406	?		
ω	76	262	3095	R	
$-\omega$	83	98	3398	$\dfrac{\left(a^2b^2+a^2c^2+b^2c^2\right)R}{a^4+b^4+c^4+a^2c^2+a^2b^2+b^2c^2}$	
$\omega/2$	?	?	511	∞	*Linfty*
$-\omega/2$	1676	1677	182	$\dfrac{R}{2\,\cos(\omega)}$	1° *Lemoine*
$\pi/4+\omega/2$	2010	2009	39	$R\sin(\omega)$	*Gallatly*
$-\pi/4-\omega/2$	?	?	32	$\dfrac{abc\left(b^2+a^2+c^2\right)}{2\,\left(b^4+c^4+a^4\right)\cos(\omega)}$	

Table 10.4: Similicenters on Kiepert RH

Chapter 11

Pencils of Cycles in the Triangle Plane

In Chapter 5, orthogonality of lines has been reduced to polarity wrt operator $\boxed{\mathcal{M}}$. Here, the same treatment will be applied to cycles, i.e. the family of all curves that are either a circle or a line. This leads to a fundamental quadric $\mathcal{Q}$ in space $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$. Finite points on the quadric can be interpreted as representatives of ordinary points from the triangle plane. Points outside this quadric are representatives of cycles, while points inside are assigned to imaginary circles.

When dealing with tangency of cycles, the best description is given by a Lie sphere, embedded into a 5D space and obtained by a double coating of the former 4D space (oriented cycles). A special attention is devoted to objects at infinity –especially the umbilics– since all these projective spaces are tailored to implement the Poncelet's continuity principle.

Efficient notations are powerful, poor notations can be confusing. In the opinion of the author, this kind of problem arises when studying pencils and bundles of circles in the context of the Triangle Geometry. On the one hand, the usual context for studying pencils and bundles of circles is Riemann sphere $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$ where $(z_1, z_2) \simeq (\lambda z_1, \lambda z_2)$ for any non-zero $\lambda \in \mathbb{C}$ (Poncelet, 1822, 1865). On the other hand, a Triangle point lives in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$ i.e. is described as $x : y : z$ in a projective space where $x : y : z = kx : ky : kz$ for any non-zero $k \in \mathbb{R}$.

Obviously, both points of view are reducing to the same elementary Cartesian coordinates when restricted to the finite domain. But they are conflicting *where they are the most useful*, i.e. where they are implementing the continuity principle for objects at infinity. An ordinary line must be completed in a way or another to become a "circle with infinite radius" and having a clear definition of this completion is required in order to unify the three concepts of circle ($0 < \rho < \infty$), point ($\rho = 0$) and line ($\rho = \infty$) into a single concept of cycle.

In the Riemann sphere $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$, there exists only one point at infinity (noted ∞). In this context, a "circle with infinite radius" is an ordinary line Δ completed by point ∞, i.e. $\overline{\Delta} = \Delta \cup \{\infty\}$, while point-circles are either circles with radius 0 around an ordinary point or $\{\infty\}$.

In the Triangle Plane $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$, there exists a whole line $\mathcal{L}_\infty$ of points at infinity, verifying $x + y + z = 0$. In this document, barycentrics are used. Using trilinears would only change some formulae, but not the very nature of the underlying space. In this context, the barycentric equation of an ordinary circle leads to define a cycle as a second degree curve (a conic) that goes through the umbilics $\Omega^\pm$. Therefore, the equation of a completed line becomes $(x + y + z)(ux + vy + wz) = 0$ so that we must define $\overline{\Delta}$ as $\Delta \cup \mathcal{L}_\infty$, while the role of $\{\infty\}$ in $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$ is played now by the horizon circle $\mathcal{C}_\infty$ defined by $(x + y + z)^2 = 0$, i.e. as an object having the same points as $\mathcal{L}_\infty$ but each of them counted twice.

Notation 11.0.1. In this section, letters used are intended to denote : P, Q some flat true points in the Triangle Plane, X a Kimberling-named triangle center ; Γ the circumcircle (and nothing else), Ω a cycle ; U the representative of a circle-point ; V the representative of a cycle in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$; Y the representative of an oriented cycle in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^5\right)$; G a Gram matrix with diagonal w^2 and non diagonal W coefficients.

Notation 11.0.2. To design a circle known by a pair center/radius, parentheses will be used, leading to $\Gamma = (X_3,\, R)$. Using parentheses around a single Roman letter –e.g. (P)– will be reserved to denote circle $(P,\, 0)$ i.e. the circle whose unique real point is P itself.

11.1 Cycles and representatives

Requiring that four points are on the same circle leads to Proposition 10.1.4, i.e. to equation :

$$\det_{i=1}^{i=4}\left[\frac{1}{p_i+q_i+r_i}\left(q_ir_ia^2+p_ir_ib^2+p_iq_ic^2\right),\ p_i,\ q_i,\ r_i\right]=0$$

But, conversely, this equation only implies that our four points are on the same circle or on the same straight line. To summarize both situations under a single concept, we introduce :

Definition 11.1.1. The **cycle** Ω associated with the **representative** $V\in\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ is is the locus of points $X=x:y:z\in\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$ that satisfies equation :

$$V=\left(\begin{array}{c}u\\v\\w\\t\end{array}\right)\quad:\quad\Gamma\left(x,y,z\right)-\left(x+y+z\right)\left(ux+vy+wz\right)\tag{11.1}$$

For example, the representative of circumcircle Γ is $0:0:0:1$.

Remark 11.1.2. When $t\neq0$, cycle Ω is the (ordinary) circle whose standardized equation is :

$$\Gamma\left(x,y,z\right)-\left(x+y+z\right)\left(\frac{u}{t}\,x+\frac{v}{t}\,y+\frac{w}{t}\,z\right)\tag{11.2}$$

Cycle $\mathcal{C}_\infty$ represented by $1:1:1:0$ has to be understood as the line at infinity $\mathcal{L}_\infty$ described twice, and will be called the **horizon circle**. Otherwise, the cycle represented by $u:v:w:0$ is the union of an ordinary line and $\mathcal{L}_\infty$, and will be called a completed line.

Theorem 11.1.3. *Let $P=p:q:r$ be a finite point. Then barycentric equation of circle $(P,\,\omega)$ is :*

$$\widetilde{t}\left(a^2\,yz+b^2\,zx+c^2\,xy\right)+\widetilde{t}\,\omega^2\,\left(x+y+z\right)^2-\left(\widetilde{u}x+\widetilde{v}y+\widetilde{x}z\right)\left(x+y+z\right)=0\tag{11.3}$$

where the 4-tuple $\left(\widetilde{u},\widetilde{v},\widetilde{w},\widetilde{t}\right)$ is defined by :

$$\begin{array}{ccl}\widetilde{u}&=&c^2q^2+b^2r^2+\left(b^2+c^2-a^2\right)qr\\\widetilde{v}&=&a^2r^2+c^2p^2+\left(c^2+a^2-b^2\right)rp\\\widetilde{w}&=&b^2p^2+a^2q^2+\left(b^2+a^2-c^2\right)pq\\\widetilde{t}&=&\left(p+q+r\right)^2\end{array}\tag{11.4}$$

Proof. Equations (11.3)(11.4) are nothing but (10.4). ☐

Remark 11.1.4. Assuming that representatives are living in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ has many advantages. The top one could be to enforce the fact that a representative is not a point in the Triangle Plane. Indeed, the triple $(\widehat{u},\,\widehat{v},\,\widehat{w})$ appearing in the standardized equation (11.2) is ***not*** defined up to a proportionality factor. The same remark applies to the so-called "circle function" $(\widehat{u}\div bc,\ \widehat{v}\div ca,\ \widehat{w}\div ab)\in\mathbb{R}^3$ that appears when using trilinears as in Weisstein (1999-2009).

11.2 Umbilics

Lemma 11.2.1. *We have $\boxed{OrtO}\,.\,\boxed{OrtO}\boxed{W}=-\boxed{W}$. Therefore, restricted to $\mathcal{V}$, $\boxed{OrtO}^2$ is nothing but an half turn. Multplied by $\boxed{Pyth}$, this leads to $\boxed{OrtO}^3+\boxed{OrtO}=0$, so that eigenvalues of $\boxed{OrtO}$ are $0,\,+i,\,-i$.*

Definition 11.2.2. Umbilics. Let $U\in\mathcal{L}_\infty$ be a point at infinity, .i.e a point such that $u+v+w=0$. The associated umbilics are the complex points :

$$\Omega^+\simeq\left(\boxed{1}-i\,\boxed{OrtO}\right).U\quad;\quad\Omega^-\simeq\left(\boxed{1}+i\,\boxed{OrtO}\right).U$$

Caveat : signs are crossed ! A possible choice is :

$$\begin{aligned} \Omega^{+} & \simeq 4SX_{512} - i\,X_{511} \\ & \simeq 4S\begin{pmatrix} a^2\left(b^2-c^2\right) \\ b^2\left(c^2-a^2\right) \\ c^2\left(a^2-b^2\right)\end{pmatrix} \pm i\begin{pmatrix} a^2\left(a^2\left(b^2+c^2\right)-b^4-c^4\right) \\ b^2\left(b^2\left(c^2+a^2\right)-c^4-a^4\right) \\ c^2\left(c^2\left(a^2+b^2\right)-a^4-b^4\right)\end{pmatrix} \end{aligned} \tag{11.5}$$

where i is the imaginary unit and S the area of the triangle. Another choice is :

$$\Omega^{+} \simeq \begin{pmatrix} S_b - 2\,iS \\ S_a + 2\,iS \\ -c^2 \end{pmatrix} \tag{11.6}$$

This is not a symmetrical expression, but computations are easier.

Proposition 11.2.3. *When seen as elements of $\mathcal{V}_{\mathbb{C}}$, all $\Omega^{\pm}$ are eigenvectors of operator $\boxed{OrtO}$, with eigenvalues (respectively) $\pm i$ and therefore belong to the light cone. When seen as elements of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$, points Ω^{+} and Ω^{-} are now independent of the choice of U, and are the fixed points of the orthopoint transform. They both belong to the circumcircle, and are isogonal conjugates of each other.*

Proof. See Postnikov (1982, 1986) for better insights on real-complex spaces. Property $\mathcal{L}_{\infty}\,.\,\Omega = 0$ is obvious. Umbilics are eigenvectors because of

$$\boxed{OrtO}\,.\,\Omega^{+} = \boxed{OrtO}\,.\,\left(\boxed{1} - i\,\boxed{OrtO}\right).U = \left(\boxed{OrtO} + i\right).U = +i\Omega^{+}$$

Since eigenvalues of $\boxed{OrtO}$ are simple, the $\mathbb{C}$-dimension of eigenspaces is one, and uniqueness in $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ follows. For this reason, points $\Omega^{\pm}$ are also called the "circular points at infinity". Finally, intersection of circumcircle and the infinity line must be invariant by isogonal conjugacy, while $\Omega^{+} *_b \Omega^{+}$ cannot be real, even up to a complex proportionality factor. □

Proposition 11.2.4. *A circle is a conic that goes through the umbilics. Above all, choosing the umbilics is deciding which of the circum ellipses is **the** circumcircle.*

Proof. By definition, umbilics are the (non real) points where line at infinity intersects the circumcircle. By (10.3), these points belong to any circle. For the converse, consider the values taken by x^2, y^2, z^2, xy, yz, zx at A, B, C together with both umbilics. This gives a 5×6 matrix whose rank is 5 : the first three lines are $1, 0, 0, 0, 0, 0$ etc. and it remains only to show that rank of submatrix 4..5,4..6 is two. A direct inspection shows that a critical factors are $a^2+b^2-c^2$ (straight angle, that can occur only once) and $a^4+b^4+c^4-b^2c^2-a^2b^2-a^2c^2$ (condition of equilaterality). In such a case, the property remains when umbilics are written as $1 : j : j^2$ and $1 : j^2 : j$. □

Remark 11.2.5. When the umbilics are given, the Euclidean structure of the Triangle Plane is known. From $\Omega^{+} *_b \Omega^{-} = a^2 : b^2 : c^2$, the $\boxed{Pyth}$ matrix is known (up to the value of R), while the orthopoint transform, and its matrix $\boxed{OrtO}$ is characterized by its diagonal shape $(0, +i, -i)$ in the X(3), Ω^{+}, Ω^{-} basis.

11.3 Pencils of cycles

Definition 11.3.1. Pencil. When Ω_1, Ω_2 are distinct cycles (with non proportional representatives), all curves $\lambda_1\Omega_1 + \lambda_2\Omega_2 = 0$, where $(\lambda_1, \lambda_2) \neq (0, 0)$, are cycles and the set of all these cycles is called the pencil generated by Ω_1, Ω_2. It is clear that representatives of the cycles of a given pencil are on the same projective line in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ –called the representative of the pencil.

Example 11.3.2. Formula (11.2) describes circle Ω as a member of the pencil generated by the circumcircle and a completed line. Therefore, the ordinary line $ux + vy + wz = 0$ is the radical axis Δ of both circles Ω and Γ. That's another way to see that knowing $u : v : w$ is not enough to determine a circle.

Example 11.3.3. Formula (11.3) describe circle (P,ω) as a member of the pencil generated by the point-circle (P) and the horizon circle, i.e. the pencil of all circles concentric with $(P,0)$. Here the horizon circle $\mathcal{C}_\infty$ is understand as a circle "whose center is everywhere and circumference nowhere" (Empedocles). This leads to :

$$u:v:w:t=\left(\widetilde{u}-\omega^2\right):\left(\widetilde{v}-\omega^2\right):\left(\widetilde{w}-\omega^2\right):\widetilde{t} \tag{11.7}$$

Definition 11.3.4. The point representative of P is another name for the representative of the point-circle (P), as given by (11.4). Points at infinity are excluded from this definition.

Remark 11.3.5. Here again, the triple $u:v:w$ is not sufficient to specify P, and $u:v:w:t$ must be used. It can be checked that representative is well specified, i.e. doesn't depends on whichever triple (kp,kq,kr) is chosen as barycentrics of point P.

Example 11.3.6. Here are some point representatives :

P	u	v	w	t
$1:0:0$	0	c^2	b^2	1
$0:1:1$	$2\,b^2+2\,c^2-a^2$	a^2	a^2	4
1	$bc\,(b+c-a)$	$ac\,(c+a-b)$	$ab\,(b+a-c)$	$a+b+c$
2	$2\,b^2+2\,c^2-a^2$	$2\,a^2+2\,c^2-b^2$	$2\,b^2+2\,a^2-c^2$	9
3	R^2	R^2	R^2	1
4	$R^2a^2\left(b^2+c^2-a^2\right)^2$	$R^2b^2\left(c^2+a^2-b^2\right)^2$	$R^2c^2\left(a^2+b^2-c^2\right)^2$	$a^2b^2c^2$
6	$b^2c^2\left(2\,b^2{+}2\,c^2{-}a^2\right)$	$a^2c^2\left(2\,c^2{+}2\,a^2{-}b^2\right)$	$a^2b^2\left(2\,a^2{+}2\,b^2{-}c^2\right)$	$\left(a^2{+}b^2{+}c^2\right)^2$
∞	1	1	1	0

The fact that formula (11.4) *would* give $1:1:1:0$ for each point on $\mathcal{L}_\infty$ is the reason of their exclusion from the definition of the point representatives.

11.4 The fundamental quadric

Proposition 11.4.1. *Any point representative $U=u:v:w:t$ belongs to the quadric $\mathcal{Q}$:*

$$\sum a^2u^2-\sum\left(b^2+c^2-a^2\right)\left(vw+a^2ut\right)+a^2b^2c^2\,t^2 \quad=\quad 0 \tag{11.8}$$

Using Conway symbols $S_a=\left(b^2+c^2-a^2\right)/2$ etc, quadric $\mathcal{Q}$ can be rewritten as ${}^tU\boxed{\mathcal{Q}}U=0$ where :

$$\boxed{\mathcal{Q}}=\frac{1}{16S^2}\left[\begin{array}{cccc} a^2 & -S_c & -S_b & -a^2S_a\\ -S_c & b^2 & -S_a & -b^2S_b\\ -S_b & -S_a & c^2 & -c^2S_c\\ -a^2S_a & -b^2S_b & -c^2S_c & a^2b^2c^2 \end{array}\right] \tag{11.9}$$

Proof. Condition (11.8) appears when trying to invert (11.4). After what, a simple substitution proves the result. Normalizing factor $1/16S^2$ will be introduced in what follows. $\square$

Definition 11.4.2. We define *Sirius* as the point $1:1:1:0\in\mathbb{P}_\mathbb{R}\left(\mathbb{R}^4\right)$. This is the representative of the locus $(x+y+z)^2=0$, i.e. the horizon circle $\mathcal{C}_\infty$. Using a star to denote $1:1:1:0$ is from Kimberling (1998-2010). Using that specific star is from (Voltaire, 1752). It is clear that Sirius belongs to $\mathcal{Q}$.

Proposition 11.4.3. *Point Sirius is the only (real) point at infinity of the quadric $\mathcal{Q}$. Therefore, $\mathcal{Q}$ is a paraboloid.*

Proof. Substitute $t=0$, then compute the discriminant with respect to w and obtain $-(u-v)^2a^2b^2c^2/R^2$. This requires $u=v$, etc. $\square$

Remark 11.4.4. In the usual $\mathbb{P}_\mathbb{C}\left(\mathbb{C}^2\right)$ model, $\mathcal{L}_\infty$ is "in the South plane" while the horizon circle $\mathcal{C}_\infty$ is nothing but the point-circle $\{\infty\}$.

Proposition 11.4.5. *A point $V = u : v : w : t$ is the representative of a (real) cycle if only if V is outside of $\mathcal{Q}$ (i.e. on the same side as $0:0:0:1$) characterized by $\mathcal{Q}(u, v, w, t) \geq 0$ when (11.8) is used.*

Proof. Obvious from (11.7), that states that representative of (P, ω) is "below" representative of (P), while representatives of completed lines are at infinity in $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$ and therefore outside of paraboloid $\mathcal{Q}$. □

Theorem 11.4.6. *Consider two cycles Ω_1, Ω_2 with representatives V_1, V_2. When V_2 belongs to the polar plane of point V_1 wrt the fundamental quadric then cycles Ω_1 and Ω_2 are orthogonal –and conversely.*

Computed Proof. Begin with two circles. Write representative V_j as in (11.7) from representative U_j of point-circle (P_j), compute ${}^tV_1 . \boxed{\mathcal{Q}} . V_2$ and –using (5.11)– obtain :

$$ {}^tV_1 . \boxed{\mathcal{Q}} . V_2 = \frac{1}{2}\left(\omega_1^2 + \omega_2^2 - |P_1P_2|^2\right) \times (p_2+q_2+r_2)^2 (p_1+q_1+r_1)^2 \tag{11.10} $$

Compute now ${}^tV_1 . \boxed{\mathcal{Q}} . V_3$ where $V_3 = u_3 : v_3 : w_3 : 0$ and obtain :

$$ {}^tV_1 . \boxed{\mathcal{Q}} . V_3 = -\frac{1}{2}(p_1u_3 + q_1v_3 + r_1w_3) \times (p_1+q_1+r_1) \tag{11.11} $$

In both cases, the result is the orthogonality condition times a non vanishing factor. Finally, when the representatives of two lines are involved, the conclusion follows directly from the properties of the orthopoint transform. □

Corollary 11.4.7. *The locus of the representatives of the points of a given cycle Ω is the intersection between $\mathcal{Q}$ and the polar plane –wrt $\mathcal{Q}$– of the representative of Ω.*

Proof. By definition, point P belongs to cycle Ω if and only if Ω and (P) are orthogonal. □

Theorem 11.4.8. ***Back to barycentrics***. *Let $V = u : v : w : t \in \mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$ be a representative. When $t = 0$ then either $V \simeq 1:1:1:0$ and V is Sirius, the representative of the horizon circle or $V \not\simeq 1:1:1:0$ and V represents a line. When $t \neq 0$, then V represents a circle (may be reduced to a point). The associated squared radius is given by :*

$$ \omega^2 = \frac{{}^tV \boxed{\mathcal{Q}} V}{t^2} \tag{11.12} $$

The representative of the center is obtained as $U = V + \omega^2$ Sirius $\in \mathcal{Q}$ while the center itself is $P = p : q : r$ where :

$$ p : q : r : \theta \simeq \boxed{\mathcal{Q}} . U \tag{11.13} $$

Proof. The radius formula is a corollary of the preceding theorem. Let $W = x : y : z : \tau$ be any cycle representative and $U \in \mathcal{Q}$ be the representative of point $P = p : q : r$. Then ${}^tW \boxed{\mathcal{Q}} U = 0$ implies

$$ px + qy + rz - \theta\tau = 0 \qquad \text{where } \theta = \left(qra^2 + rpb^2 + pqc^2\right) / (p+q+r) $$

so that equation (11.13) must hold for rank reason (and can be checked directly). Conversely, starting from any V and applying (11.13) and then (11.4) leads back to V. □

Theorem 11.4.9 (Classification)**.** *Pencils of cycles fall in three classes, depending on the way their representative line $\mathcal{P}$ intersects –in $\mathbb{P}_{\mathbb{R}}(\mathbb{R}^4)$– the fundamental quadric $\mathcal{Q}$.*

$\mathcal{Q}$, $\mathcal{P}$ tangent : $\mathcal{P}$ is the **tangent pencil** *of all the cycles containing a given point ω_0 and tangent at ω_0 to a line Δ_1 containing ω_0. Archetype : $\omega_0 = \infty$ and $\mathcal{P}$ is "all the lines parallel to a given line Δ_1".*

$\mathcal{Q}$, $\mathcal{P}$ secant : $\mathcal{P}$ is the **isotomic pencil** *generated by two different point-circles $\{\omega_1\}$ and $\{\omega_2\}$ (ω_i are the **limit** points of $\mathcal{P}$). Archetype : $\omega_2 = \infty$ and $\mathcal{P}$ is, apart from $\{\infty\}$, "all the circles centered at a finite point ω_1".*

$\mathcal{Q}$, $\mathcal{P}$ *disjoint :* $\mathcal{P}$ *is the* **isoptic pencil** *of all the cycles going through two different points* ω_1 *and* ω_2 *(the* ***base*** *points). Archetype :* $\omega_2 = \infty$ *and* P *is "all the lines through a finite point* ω_1*".*

When $\mathcal{P}$ *is a tangent pencil, so is* $\mathcal{P}^{\perp}$ *(using* ω_0 *and* $\Delta_1^{\perp}$ *orthogonal to* Δ_1 *at* ω_0*). When* $\mathcal{P}$ *is isoptic* (ω_1, ω_2) *then* $\mathcal{P}^{\perp}$ *is isotomic* (ω_1, ω_2) *and conversely. In all cases, representative of* $\mathcal{P}$ *and* $\mathcal{P}^{\perp}$ *are conjugate lines wrt* $\mathcal{Q}$*.*

Proof. Everything goes as in (Pedoe, 1970) –using another paraboloid– or (Douillet, 2009) –using a sphere. The only striking thing is that the usual point at infinity of the complex plane, namely $\infty \in \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$, has to be replaced by the horizon circle $\mathcal{C}_{\infty} : (x+y+z)^2 = 0$. □

Proposition 11.4.10. *A pencil of cycles that contains two lines is a pencil of lines. A concentric pencil contains the horizon cycle. All other pencils (i.e. all the non archetypal pencils) contain exactly one straight line (the radical axis of the pencil).*

Proof. The representative of $\mathcal{P}$ ever intersects the plane at infinity of $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$. □

Proposition 11.4.11. *Let* $P = p : q : r$ *be a point in the Triangle Plane. Define its shadow in the Triangle Plane as point* $S = u : v : w$ *where* u, v, w *are defined in* (11.4)*. Then* S *is not outside the inconic* $IC(X_{76})$*. Any point on the border of* $IC\left(X_{76}\right)$ *is the shadow of exactly one point on the circumcircle, while a point inside* $IC\left(X_{76}\right)$ *–except from* X_2*– is the shadow of exactly two points. Moreover, these points are inverse in the circumcircle.*

Remark 11.4.12. Figure 11.1 shows the shadows of all the named points in ETC (Kimberling, 1998-2010), using the standard values $a = 6$, $b = 9$, $c = 13$. One can see two lines of points : $L\left(X_2,\, X_6\right)$ and $L\left(X_2,\, X_{39}\right)$ containing the shadows of points from $L\left(X_3,\, X_2\right)$ –Euler line– and $L\left(X_3,\, X_6\right)$ –Brocard axis– respectively.

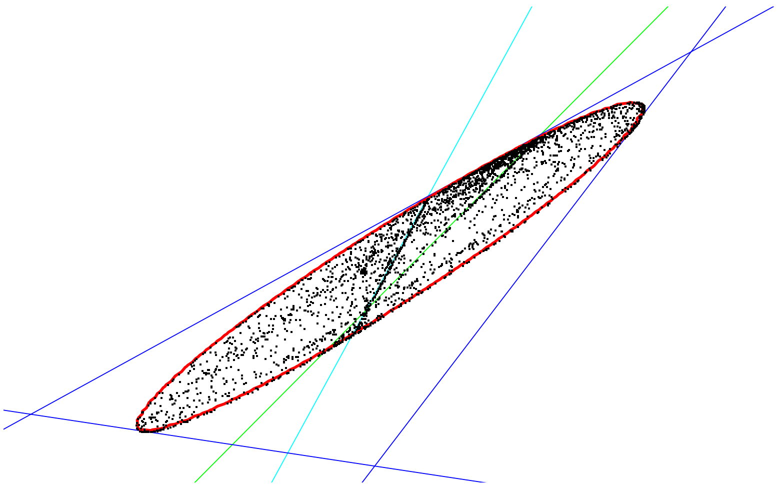


Figure 11.1: No point-shadow fall outside of the IC(X76) inconic

Proof of Proposition 11.4.11 . The locus of representatives of the points P_0 that belongs to Γ is the intersection of quadric $\mathcal{Q}$ and the polar plane $\prod$ of $0:0:0:1$, namely :

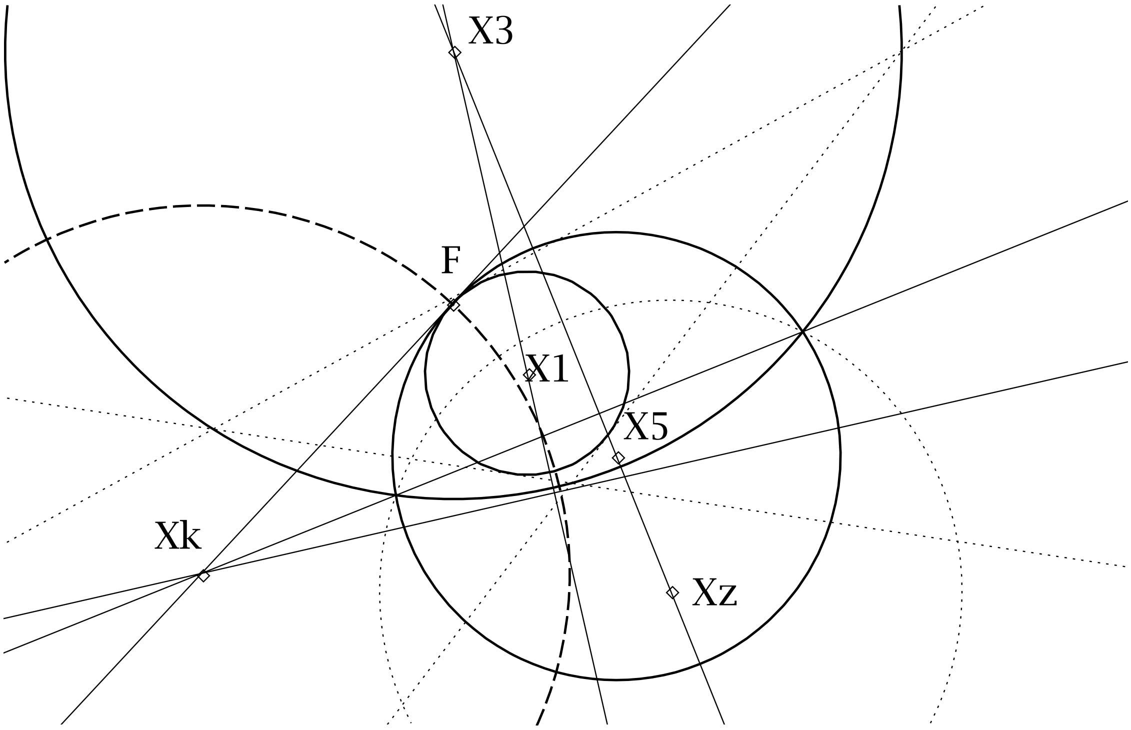
$$ua^2\left(b^2+c^2-a^2\right)+vb^2\left(c^2+a^2-b^2\right)+wc^2\left(a^2+b^2-c^2\right)-2\,ta^2b^2c^2=0$$

Extracting t and substituting in $\mathcal{Q}$ leads (apart from a constant non-zero factor) to :

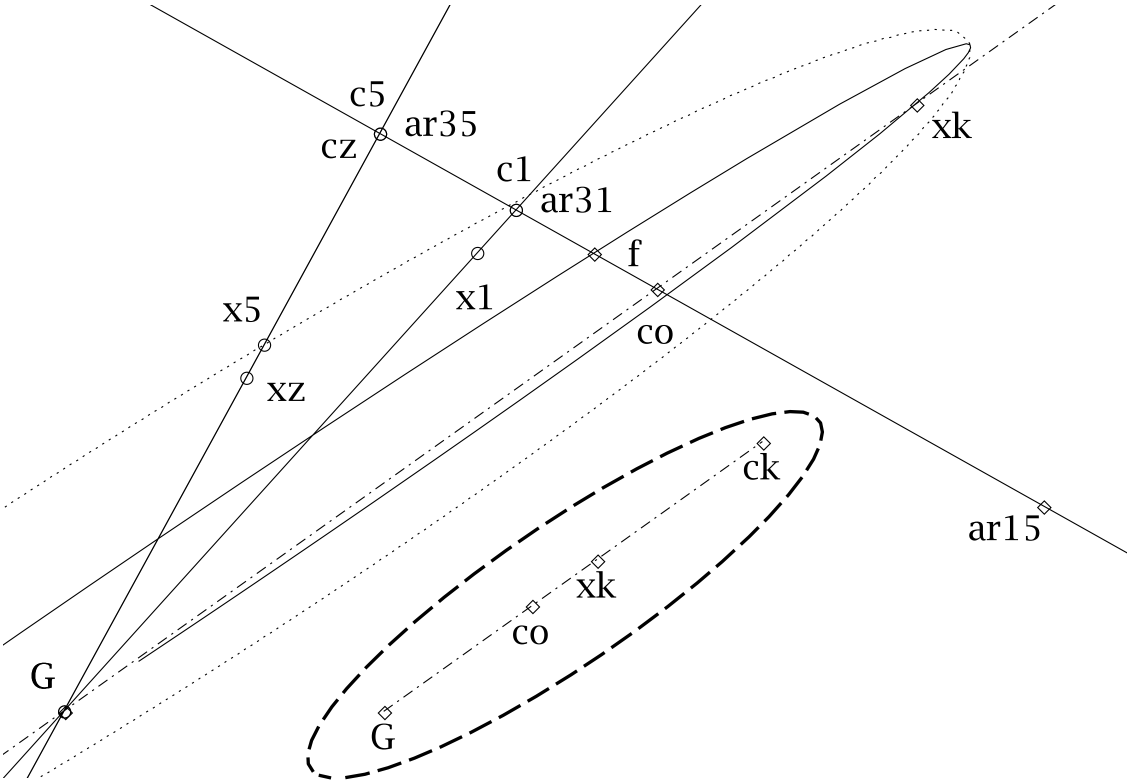
$$u^2a^4+b^4v^2+c^4w^2-2\,uva^2b^2-2\,vwb^2c^2-2\,wuc^2a^2=0$$

i.e. the equation of $IC\left(X_{76}\right)$.

When two points P_1, P_2 in the Triangle Plane share the same shadow, then points U_1, U_2 and $0:0:0:1$ are collinear in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ so that cycles (P_1), (P_2) and Γ belongs to the same pencil. Therefore P_1, P_2 are inverse in the circumcircle. Moreover $U_0 \doteq U_1U_2 \cap \prod$ is inside $IC\left(X_{76}\right)$ –otherwise U_0 would be the representative of a real circle belonging to pencil $(P_1), (P_2)$ and orthogonal to Γ. □



(a) lines and circles



(b) shadows

Figure 11.2: Euler pencil and incircle

Theorem 11.4.13. *Let Ω_0 be a fixed cycle with representative V_0 and Ω_1 another cycle with representative V_1. Assume that Ω_0 is not a point-circle and call $\widehat{V_1}$ the intersection of line V_0V_1 with the polar plane of Ω_0. Construct V_3 such that division $V_0, \widehat{V_1}, V_1, V_3$ is harmonic. Then V_3 represents the cycle Ω_3 inverse of Ω_1 wrt cycle Ω_0 (inverse wrt a straight line is the ordinary symmetry wrt this line) and is given by :*

$$V_3 \simeq \sigma(V_1) \doteq V_1 - 2\,\frac{{}^tV_1 \boxed{\mathcal{Q}} V_0}{{}^tV_0 \boxed{\mathcal{Q}} V_0}\, V_0 \tag{11.14}$$

Moreover, when V_2 is yet another cycle representative, we have the conservation law :

$${}^t\sigma(V_1) \boxed{\mathcal{Q}}\, \sigma(V_2) = {}^tV_1 \boxed{\mathcal{Q}} V_2 \tag{11.15}$$

Proof. Write $\widehat{V_1}$ as $\alpha_1 V_0 + V_1$ in ${}^tV_0 \boxed{\mathcal{Q}} \widehat{V_1} = 0$ and then obtain V_3 as $2\alpha_1 V_0 + V_1$ since division $(\infty, 1, 0, 2)$ is harmonic. Equation (11.15) is obvious from (10.8), and shows that σ preserves orthogonality. Moreover, (10.8) shows that cycles orthogonal to Ω_0 are invariant while cycles concentric with Ω_0 are transformed into cycles concentric with Ω_0 : all together, this proves that σ is the inversion wrt cycle Ω_0. □

Proposition 11.4.14. *Points inside of $\mathcal{Q}$ are representative of imaginary circles (real center, imaginary radius). The reason to imagine such circles is that inversion wrt such a circle is a real transform. Moreover a real cycle Ω is orthogonal to $(X, i\omega)$ when Ω cuts (X, ω) along a diameter.*

Proof. Straightforward computation. □

Proposition 11.4.15. *It exists exactly one cycle Ω_4 simultaneously orthogonal to three cycles Ω_j (j=1,2,3) that don't belong to the same pencil. Its representative can be computed as :*

$$V_4 = Adjoint\,(\mathcal{Q}) \cdot {}^t(V_1 \wedge V_2 \wedge V_3) \tag{11.16}$$

where $\wedge$ is the universal factorization of $\det(V_1, V_2, V_3, V)$*, aka the row of cofactors involving the V_j (j=1,2,3).*

Remark 11.4.16. The nature of Ω_4, real, point or imaginary fixes the nature of the bundle defined by $\Omega_1, \Omega_2, \Omega_3$.

11.5 Euler pencil and incircle

Consider $\mathcal{C}_1 = (X_1, r)$, $\mathcal{C}_3 = (X_3, R)$, $\mathcal{C}_5 = (X_5, R/2)$ and $\mathcal{C}_z = (X_z = X_{389}, |GH|/2)$ i.e., respectively, the in-, circum- nine points and orthocentroidal circles (Figure 11.2a). Let U_j, V_j, x_j, c_j be the respective representatives of centers and circles, together with their respective shadows (Figure 11.2b). Then :

1. Circles (X_1), $\mathcal{C}_1$, $\mathcal{C}_\infty$ are concentric so that U_1, V_1, *Sirius* are aligned and therefore x_1, c_1, G are aligned too. The same happens for $j = 5$ and $j = z$.

2. Cycles $\mathcal{C}_3$, $\mathcal{C}_5$, $\mathcal{C}_z$ belong to the same (Euler) pencil, together with their radical axis $AR_{3,5}$, so that representatives V_3, V_5, V_z, $V_{3,5}$ are aligned and therefore c_3, c_5, c_z, $ar_{3,5}$ are aligned too. Since c_3 is "far below the paper sheet", we have $c_5 = c_z = ar_{3,5}$. For the same reason, $c_1 = ar_{3,1}$.

3. Circles $\mathcal{C}_1$ and $\mathcal{C}_5$ are tangent at $F \doteq X_{11}$, the Feuerbach point and cycles (F), $\mathcal{C}_1$, $\mathcal{C}_5$ belong to the same pencil, together with their common tangent $AR_{1,5}$. Representatives U_f, $V_1, V_5, V_{1,5}$ are aligned and so are x_f, c_1, c_5, $ar_{1,5}$.

4. Cycles $AR_{1,3}$, $AR_{1,5}$, $AR_{3,5}$ are on the same pencil (they concur in the radical center X_k) and their shadows $ar_{1,3}$, $ar_{1,5}$, $ar_{3,5}$ are aligned.

5. In fact line c_1c_5 is not representative of a specific pencil, but rather of the bundle generated by $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_5$. We have :

$$V_1 \simeq \left[\begin{array}{c} (b+c-a)^2 \\ (c+a-b)^2 \\ (a+b-c)^2 \\ 4 \end{array}\right], V_5 \simeq \left[\begin{array}{c} b^2+c^2-a^2 \\ c^2+a^2-b^2 \\ a^2+b^2-c^2 \\ 4 \end{array}\right], V_3 \simeq \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right]$$

and therefore :

$$V_k \simeq \left[\begin{array}{c} (c-b)(b+c-a)\left(b^2+c^2-a^2\right)\left(b^2+c^2-ab-ac\right) \\ (a-c)(c+a-b)\left(c^2+a^2-b^2\right)\left(c^2+a^2-bc-ba\right) \\ (b-a)(a+b-c)\left(a^2+b^2-c^2\right)\left(a^2+b^2-ca-cb\right) \\ 4\,(a-b)(b-c)(c-a)(a+b+c) \end{array}\right]$$

From V_k, well known result $X_k = X_{676}$ and obvious $r_k = |X_k X_f|$ can be re-obtained.

6. As it should be, x_k, c_k, G are aligned (small insert, at the bottom of Figure 11.2b).

7. Consider W_k at intersection between *line* $(V_k Sirius)$ and *plane* (V_1, V_3, V_5). This points represents a circle that is both concentric and orthogonal to $\mathcal{C}_k$.This circle is therefore $(X_k,\ i\,r_k)$ and its shadow co belongs to both Gx_k and c_5c_1. Moreover, division G, x_k, co, c_k is harmonic.

11.6 The Apollonius configuration

In the general case, it exists eight cycles Ω tangent to three given cycles Ω_1, Ω_2, Ω_3 (not from the same pencil). A survey of this question is Gisch and Ribando (2004), while the usual disjunction into ten cases is Wiki Contributors (2008). The best space where this Apollonius problem can be discussed is $\mathbb{P}_\mathbb{R}(\mathbb{R}^5)$ (cf Section 11.7). Nevertheless, most of the results can be formulated in $\mathbb{P}_\mathbb{R}(\mathbb{R}^4)$... and it will appear that only one situation is really special (cycles through the same point), all the other belonging to the same general case.

11.6.1 Tangent cycles in the representative space

Proposition 11.6.1. *Two cycles are tangent when their pencil line is tangent to the fundamental quadric. Therefore, the locus of the representatives of all cycles tangent to a given (real) cycle Ω represented by V (not inside $\mathcal{Q}$) is the cone whose vertex is V and that goes through $\mathcal{Q} \cap polar(V)$.*

Proof. Two tangent cycles are defining a tangent pencil ! □

Remark 11.6.2. When Ω is a point circle, its representative U belongs to the fundamental quadric, and the cone of the tangent cycles degenerates into a doubly coated plane.

Definition 11.6.3. The Gram matrix $\mathrm{G}_{p,q,\cdots,r}$ of X_p, X_q, $\cdots$, $X_r \in \mathbb{R}^4$ is the matrix of all the products ${}^tX_p \boxed{\mathcal{Q}} X_q$. In this context, notation $W_{pq} = {}^tX_p \boxed{\mathcal{Q}} X_q$ and $w_p^2 = {}^tX_p \boxed{\mathcal{Q}} X_p$ will be used, leading to

$$\mathrm{G}_{pq} = \left(\begin{array}{cc} w_p^2 & W_{pq} \\ W_{pq} & w_q^2 \end{array}\right) \tag{11.17}$$

Proposition 11.6.4. *Two cycles Ω_1, Ω_2 are secant, tangent or external when* $\operatorname{signum}\det \mathrm{G}_{12}$ *is (respectively) $+1$, 0 or -1.*

Theorem 11.6.5. *Special cases of the Apollonius problem are (1) cycles from the same pencil and (2) cycles through the same point (tangent bundle). Otherwise, representatives V_j of the three given cycles and their common orthogonal cycle Ω_4 form a basis that splits the problem into four pairs of solutions. One of the solutions is given by $V_0 = \sum k_j\, V_j$ where :*

$$\begin{array}{rcl} k_1 & = & (w_2\, w_3 - W_{23})\,(-w_1\, w_2\, w_3 - w_1\, W_{23} + w_2\, W_{13} + w_3\, W_{12}) \\ k_2 & = & (w_1\, w_3 - W_{13})\,(-w_1\, w_2\, w_3 + w_1\, W_{23} - w_2\, W_{13} + w_3\, W_{12}) \\ k_3 & = & (w_1\, w_2 - W_{12})\,(-w_1\, w_2\, w_3 + w_1\, W_{23} + w_2\, W_{13} - w_3\, W_{12}) \\ k_4 & = & \sqrt{-2\,(w_2\, w_3 - W_{23})\,(w_1\, w_3 - W_{13})\,(w_1\, w_2 - W_{12})\;\mathrm{G}_{123}}\,/w_4 \end{array} \tag{11.18}$$

and the others are obtained by changing k_4 into $-k_4$ (inversion through Ω_4) or changing the signs of w_1, w_2, w_3. A solution is real/imaginary or "unimaginable" (object that would have a non real center) according to the sign of k_4^2. Globally, the number of "imaginable" solutions changes when the tangency condition $\prod \mathrm{G}_{jk}$ vanishes.

Proof. When Ω_j, $j = 1, 2, 3$ is a basis of a non tangent bundle, then Ω_j, $j = 1, 2, 3, 4$ is a basis of the whole representative space. The fundamental quadratic form is described, in this basis, by matrix G_{1234} where $W_{j4} = 0$ for $j = 1, 2, 3$. Computing, in this basis, the tangency condition of Ω_0 and any of the Ω_j leads to 0. Since the w_j are defined as $\sqrt{W_{jj}}$ we have 4 choices of signs leading, due to the possibility of a global proportionality factor, to eight different values. □

11.6.2 An example: the Soddy circles

Proposition 11.6.6. ***Soddy circles*** *are three mutually, externally, tangent circles. Let A, B, C be their centers. Then the common orthogonal circle of the Soddy's is the incircle of ABC.*

Proof. Let x be the radius of circle (A), etc. We have $a = y + z$, $b = z + x$, $c = x + y$. Therefore $x = b + c - a$, etc, while the contact point of the $(B), (C)$ circles is $G_a \simeq 0 : y : z$. This is also the contact point of (I) with line BC and the conclusion follows. □

Proposition 11.6.7. *The Apollonius circles of the Soddy circles are twice each of them, a small circle (inside the intouch triangle) and an outer circle. The center of the smaller circle is called X(176), the other is called X(175). Let $\mathcal{H}_a$ be the branch of hyperbola that goes through A, G_a and has B, C as focuses. Then the three branches concur at X(176), while the other three branches of the whole hyperbolas concur at X(175).*

Proof. This is clear from $a = y + z$, etc. □

Proposition 11.6.8. *Centers and radiuses of the Soddy circles are given by :*

$$\begin{array}{rcl} nX(175) & = & \dfrac{4R+\rho}{4R+\rho-2s}\, nX(7) + \dfrac{-2s}{4R+\rho-2s}\, nX(1) \\ nX(176) & = & \dfrac{4R+\rho}{4R+\rho+2s}\, nX(7) + \dfrac{2s}{4R+\rho+2s}\, nX(1) \end{array}$$

$$\frac{1}{\rho_6} - \frac{1}{\rho_5} = \frac{4}{\rho}\;;\;\frac{1}{\rho_6} + \frac{1}{\rho_5} = \frac{8R+2\rho}{s\,\rho}$$

Proof. Using (11.7), the representatives of circles $\Omega_1 \cdots \Omega_4$are :

$$\left(\begin{array}{cccc} (b+c-a)^2 & (b-3\,c-a)\,(b+c-a) & (c-3\,b-a)\,(b+c-a) & (b+c-a)^2 \\ (a-3c-b)\,(c+a-b) & (c+a-b)^2 & (c-3\,a-b)\,(c+a-b) & (c+a-b)^2 \\ (a-3b-c)\,(a+b-c) & (b-3\,a-c)\,(a+b-c) & (a+b-c)^2 & (a+b-c)^2 \\ -4 & -4 & -4 & 4 \end{array}\right)$$

Their Gram matrix is :

$$\left(\begin{array}{cccc} (b+c-a)^2 & -(c+a-b)\,(b+c-a) & -(a+b-c)\,(b+c-a) & 0 \\ -(b+c-a)\,(c+a-b) & (c+a-b)^2 & -(a+b-c)\,(c+a-b) & 0 \\ -(b+c-a)\,(a+b-c) & -(c+a-b)\,(a+b-c) & (a+b-c)^2 & 0 \\ 0 & 0 & 0 & \dfrac{16\,S^2}{(b+a+c)^2} \end{array}\right)$$

Then (11.18) gives the decomposition of the Soddy's circles on the Ω basis. We have :

$$K \simeq \left(\begin{array}{c} (b+c-a)^{-1} \\ (c+a-b)^{-1} \\ (a+b-c)^{-1} \\ \dfrac{b+a+c}{2\,S} \end{array}\right)$$

and the conclusion follows (here $a+b+c=2s$). □

11.6.3 An example: the not so Soddy circles

Proposition 11.6.9. *The **not so Soddy circles** of a triangle are the circles centered at A with radius $a=BC$, etc. Their common orthogonal circle is the Longchamps circle λ (i.e. the conjugate circle of the antimedial triangle).*

Proof. The circle γ_a, centered at A with radius a is described by :

$$yza^2+b^2zx+c^2yx+(x+y+z)\left(a^2x+y\left(a^2-c^2\right)+z\left(a^2-b^2\right)\right)=0$$

Its intersections with the circumcircle are :

$$P_b \simeq a^2-c^2 : -b^2 : c^2-a^2 \quad \textit{and} \quad P_c \simeq a^2-b^2 : b^2-a^2 : -c^2$$

while its intersections with γ_b (resp. γ_c) are P_c and $C' \simeq 1:1:-1$ (resp. P_b and $B' \simeq 1:-1:1$). Points A', B', C' are on circle $(H, 2R)$ and form the antimedial triangle. Lines P_aA' are the altitudes of $A'B'C'$ and concur at $L \doteq H'$ =X(20). But they are also the radical axes of our circles. The radius r_L of the orthogonal circle is obtained by :

$$r_L^2=|AL|^2-a^2,\ \text{etc} = \frac{-S_aS_bS_c}{S^2}=4\,(2R+\rho+s)\,(2R+\rho-s)$$

□

Proposition 11.6.10. *The Apollonius circles of the not so Soddy circles are obtained by extraversions (i.e $a \mapsto -a$) from their central versions. These central circles are centered at the Soddy points X(175) and X(176).*

Proof. The representatives and the Gram matrix of γ_a, γ_b, γ_c, λ are :

$$\left(\begin{array}{cccc} -a^2 & c^2-b^2 & b^2-c^2 & a^2 \\ c^2-a^2 & -b^2 & a^2-c^2 & b^2 \\ b^2-a^2 & a^2-b^2 & -c^2 & c^2 \\ 1 & 1 & 1 & 1 \end{array}\right), \left(\begin{array}{cccc} a^2 & S_c & S_b & 0 \\ S_c & b^2 & S_a & 0 \\ S_b & S_a & c^2 & 0 \\ 0 & 0 & 0 & -S_a\,S_b\,S_c \div S^2 \end{array}\right)$$

Changing $w_1 \mapsto -w_1$ is $a \mapsto -a$, proving the extraversions. Formula(11.18) gives the coefficients :

$$\left(\begin{array}{c} (bc-S_a)\,(-bca-aS_a+bS_b+cS_c) \\ (ca-S_b)\,(-bca-bS_b+cS_c+aS_a) \\ (ab-S_c)\,(-bca-cS_c+aS_a+bS_b) \\ 16\,S^3 \div (a+b+c) \end{array}\right)$$

Organizing the obtained equations, we have :

$$\begin{array}{rcl} \Gamma_6 & = & \lambda+(x+y+z)\,(ax+by+cz)\times\dfrac{4s\,(2R+\rho+s)}{4R+\rho+2s} \\ \Gamma_5 & = & \lambda+(x+y+z)\,(ax+by+cz)\times\dfrac{4s\,(2R+\rho-s)}{4R+\rho-2s} \end{array}$$

This leads to the centers. Additionally, this proves that $ax+by+cz=0$ is the radical axis of the three circles. Moreover, the Soddy conic (through A,B,C, with focuses X(175), X(176) and perspector X(7)) is tangent to the Longchamps circle at the common points of λ, Γ_6, Γ_5 since we have :

$$\textit{conic}=\lambda+(ax+by+cz)^2$$

□

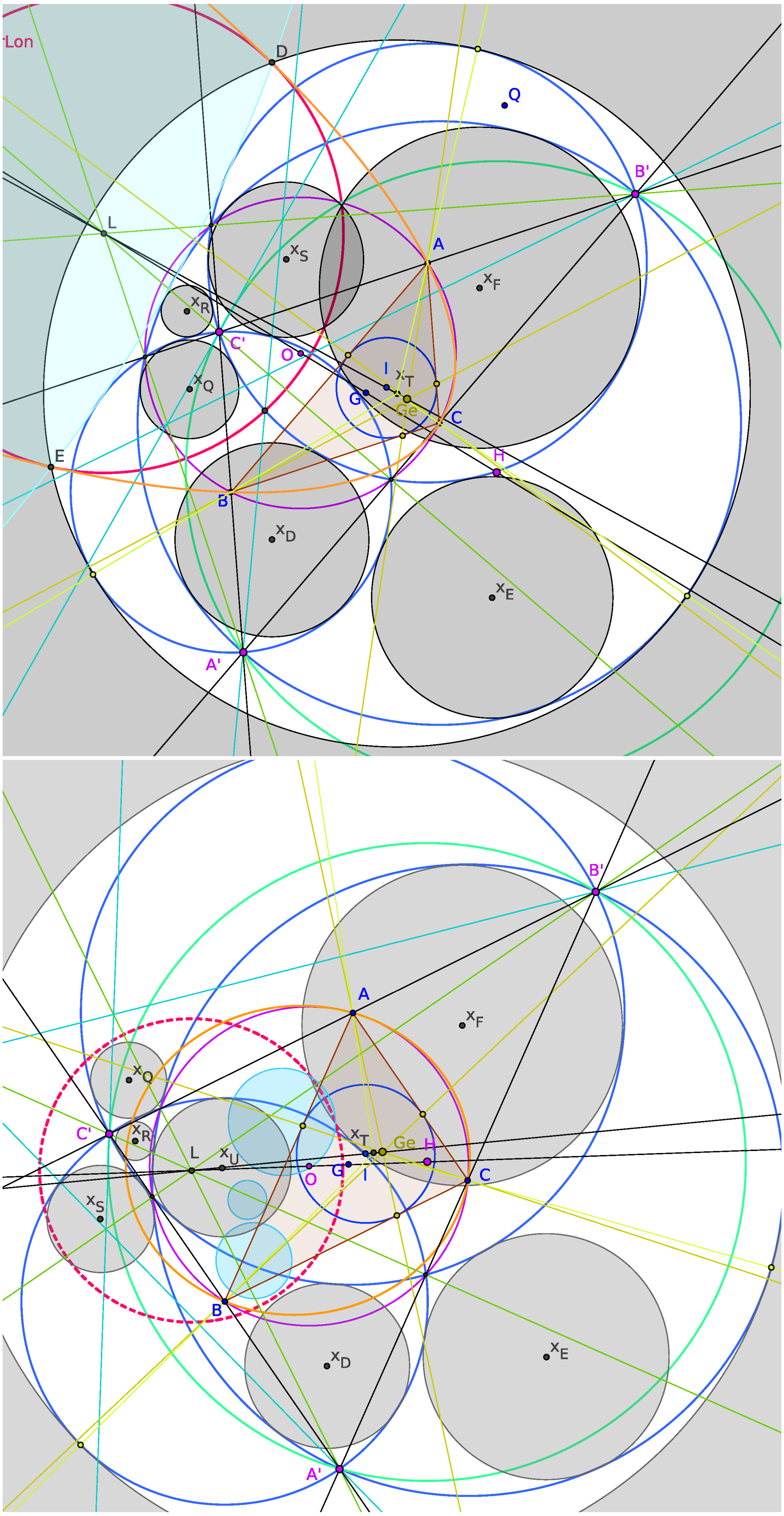


Figure 11.3: Apollonius circles of the not so Soddy configuration.

11.6.4 Yet another example : the three excircles

Taking the three excircles as $\Omega_1, \Omega_2, \Omega_3$ leads to a well known situation (Stevanovic, 2003).

1. Representative of point X_1 is :

$$U_0 = \left(\begin{array}{c} bc\,(b+c-a) \\ ca\,(c+a-b) \\ ab\,(a+b-c) \\ a+b+c \end{array} \right)$$

while radius of the incircle is :

$$r = \sqrt{\frac{(a+b-c)\,(c+a-b)\,(b+c-a)}{4\,(a+b+c)}}$$

2. Representative of the incircle, given by (11.7), is :

$$V_0 = \left(\begin{array}{c} (a-b-c)^2 \\ (a-b+c)^2 \\ (a+b-c)^2 \\ 4 \end{array} \right)$$

3. Centers, radiuses and representatives V_a, V_b, V_c of the excircles are obtained by changing one of the sidelengths into its opposite in the respective formulae for the incircle.

4. The representative of the common orthogonal circle, as computed from (11.16), is :

$$V_4 = \left(\begin{array}{c} (c+a-b)\,(a+b-c) \\ (a+b-c)\,(b+c-a) \\ (b+c-a)\,(c+a-b) \\ -4 \end{array} \right)$$

5. The radius of this circle, as computed from (11.12), is :

$$\omega_4 = \sqrt{\frac{b^2c + ab^2 + bc^2 + a^2b + ac^2 + a^2c + acb}{4\,(a+b+c)}}$$

while the representative of the center is :

$$U_4 = \left(\begin{array}{c} 2\,a\left(b^2+c^2\right) - acb + b^3 + c^3 - a^3 \\ 2\,b\left(c^2+a^2\right) - acb + c^3 + a^3 - b^3 \\ 2\,c\left(a^2+b^2\right) - acb + a^3 + b^3 - c^3 \\ 4\;(a+b+c) \end{array} \right)$$

and the center itself is :

$$b+c : a+c : a+b = X_{10}$$

6. The pairs of solutions of the Apollonius problem, as given by (11.18), are :

$$S_1 = \left(\begin{array}{c} b^2+c^2-a^2 \\ c^2+a^2-b^2 \\ a^2+b^2-c^2 \\ 4 \end{array} \right),\ S_5 = \left(\begin{array}{c} bc\,(a+b+c)\,(2\,bc + a\,(a+b+c)) \\ ca\,(a+b+c)\,(2\,ca + b\,(a+b+c)) \\ ab\,(a+b+c)\,(2\,ab + c\,(a+b+c)) \\ -4\,abc \end{array} \right)$$

$$S_2 = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right),\ S_6 = \left(\begin{array}{c} (a+b+c)\left(b^2+ab+ac+c^2\right) \\ (b+c)\,(a-b-c)\,(a+b-c) \\ (b+c)\,(a-b-c)\,(a-b+c) \\ 4\;(b+c) \end{array} \right)$$

Point S_1 is the representative of the nine-points circle, centered at X_5 while S_5 is related to the Apollonius circle, centered ad X_{970}. Points S_2, S_3, S_4 are the representatives of lines BC, CA, AB while S_6 and S_7, S_8 (obtained cyclically) are the representatives of the last three solutions.

11.6.5 The special case

Proposition 11.6.11. *Let Ω_1, Ω_2, Ω_3 be three cycles generating a bundle whose common orthogonal cycle is a point-cycle (ω_5), and ω_4 be any other point. The representative of one of the cycles tangent to Ω_1, Ω_2, Ω_3 is given by $V_0 = \sum_1^3 k_j\, V_j + 4U_4$ where :*

$$k_1 = \left(\frac{w_2 w_3 - W_{2,3}}{(w_1 w_3 - W_{1,3})\,(w_1 w_2 - W_{1,2})}\ \mathrm{G}_{1,2,3,4} - 2\ \mathrm{G}_{2,3,4} \right) \div \Delta_{2,3,4}^{1,2,3} \tag{11.19}$$

$\Delta_{2,3,4}^{1,2,3}$ is the minor obtained by deleting row 1 and column 4 in $\mathrm{G}_{1,2,3,4}$ and k_2, k_3 are obtained cyclically. Three other cycles are obtained by changing one of the w_1, w_2, w_3 into its opposite. The other solutions are four times the point cycle ω_5.

Proof. In this special case, $\mathrm{G}_{1,2,3} = 0$ and Ω_4 is chosen so that $w_4 = 0$. When assuming that Ω_1, Ω_2, Ω_3 aren't pairwise tangent, a direct substitution shows that Ω_0 is tangent to any of the given cycles. □

Example 11.6.12. Using $\Omega_1 = 1 : 0 : 0 : 0$ (representative of line BC) etc, leads to $\omega_5 = Sirius$. An efficient choice for ω_4 is any vertex. Using, for example, $U_4 = 0 : c^2 : b^2 : 1$, one re-obtains easily the in/excircles.

Example 11.6.13. Let (XYZ) be the circumcircle of triangle XYZ. The Apollonius circles relative to (ABH), (BCH), (CAH), i.e. the circles tangents to all of the three given circles, are $(H, 0)$ four times, $(H, 2R)$ once and three other circles, Ta, Tb, Tc.

Circles Ta, Tb, Tc are ever external to each other, and their common orthogonal circle To is real. Condition of (external) tangency is :

$$\left(a^2 - b^2\right)^2 - \left(a^2 + b^2\right) c^2 = 0$$

(etc) or ABC rectangular. The Apollonius circles of Ta, Tb, Tc are (ABH) etc, their inverses in To and two others. Center $X = x : y : z$ and radius ω of the first one are :

$$x = \left(b^2 + c^2 - a^2\right) \times \\ \left(a^8 - 2\left(b^2 + c^2\right) a^6 + 2\left(b^4 - b^2c^2 + c^4\right) a^4 - 2\,\left(b^2 + c^2\right)\left(b^2 - c^2\right)^2 a^2 + \left(b^2 - c^2\right)^4\right)$$

$$\omega = 2\,\frac{a^2b^2c^2R}{a^6 + b^6 + c^6 - a^2b^4 - a^4b^2 - c^2b^4 - a^4c^2 - b^2c^4 - c^4a^2 + 4\,a^2b^2c^2}$$

while the second is less simple.

11.7 Elementary properties of the Triangle Lie Sphere

Definition 11.7.1. The Triangle Lie Sphere is the locus of points $X = x : y : z : t : \tau \in \mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^5\right)$ such that ${}^tX \,.\, \boxed{\mathcal{Q}_5} \,.\, X = 0$ where fundamental matrix is defined by :

$$\boxed{\mathcal{Q}_5} = \begin{bmatrix} \boxed{\mathcal{Q}} & 0 \\ 0 & -4S^2 \end{bmatrix} = \begin{bmatrix} a^2 & -S_c & -S_b & -a^2S_a & 0 \\ -S_c & b^2 & -S_a & -b^2S_b & 0 \\ -S_b & -S_a & c^2 & -c^2S_c & 0 \\ -a^2S_a & -b^2S_b & -c^2S_c & a^2b^2c^2 & 0 \\ 0 & 0 & 0 & 0 & -4\,S^2 \end{bmatrix}$$

Proposition 11.7.2. *Using notations of Theorem 11.4.6, an element of the Lie sphere represents, when $t \neq 0$ and $\tau \neq 0$, an oriented circle :*

$$Y_1 = \left[\begin{array}{c} b^2 r_1^2 + c^2 q_1^2 + 2\, S_a q_1 r_1 - \omega_1^2 \left(p_1 + q_1 + r_1\right)^2 \\ c^2 p_1^2 + a^2 r_1^2 + 2\, S_b r_1 p_1 - \omega_1^2 \left(p_1 + q_1 + r_1\right)^2 \\ a^2 q_1^2 + b^2 p_1^2 + 2\, S_c p_1 q_1 - \omega_1^2 \left(p_1 + q_1 + r_1\right)^2 \\ \left(p_1 + q_1 + r_1\right)^2 \\ 2\, \omega_1 \left(p_1 + q_1 + r_1\right)^2 \end{array} \right]$$

(where signum of ω_1 defines the orientation) or, when $t = 0$ and $\tau \neq 0$, an oriented line :

$$Y_3 = \left[\begin{array}{c} u_3 \\ v_3 \\ w_3 \\ 0 \\ \pm \dfrac{1}{2S} \sqrt{\Delta \,.\, \mathcal{M} \,.\, {}^t\Delta} \end{array} \right]$$

–cf (7.3) for definition of $\mathcal{M}$– or, when $\tau = 0$, a non-oriented point.

Proof. All these results follow directly from Theorem 11.4.6. □

Proposition 11.7.3. *Two oriented cycles are tangent if and only if ${}^tY_j \boxed{\mathcal{Q}_5} Y_k$ vanishes.*

Proof. This result is the rationale behind the former definitions. Using notations of Theorem 11.4.6, we obtain for two circles :

$${}^tY_1 \boxed{\mathcal{Q}_5} Y_2 = -\left(\left|P_1 P_2\right|^2 - \left(\omega_1 - \omega_2\right)^2 \right) \times 8 S^2 \, \left(p_1 + q_1 + r_1\right)^2 \left(p_2 + q_2 + r_2\right)^2 \tag{11.20}$$

By continuity, the Proposition holds also when lines are involved. For the sake of completeness, one can nevertheless compute :

$${}^tY_1 \boxed{\mathcal{Q}_5} Y_3 = -\left(\frac{p_1 u_3 + q_1 v_3 + r_1 w_3}{p_1 + q_1 + r_1} + \omega_1 \frac{1}{2S} \sqrt{\Delta_3 \,.\, \mathcal{M} \,.\, {}^t\Delta_3} \right) \times 8 S^2 \, \left(p_1 + q_1 + r_1\right)^2 \tag{11.21}$$

for a circle and a line, and (after multiplication by conjugate quantity) :

$$-\left(\left(v_4 - w_4\right) u_3 + \left(w_4 - u_4\right) v_3 + \left(-v_4 + u_4\right) w_3 \right)^2 \times 8\, S^2$$

for two lines Y_3, Y_4. In the three cases, this is the condition of tangency times a non vanishing factor. □

Remark 11.7.4. When using the Lie representation, objects that don't belong to $\mathcal{Q}_5$ are meaningless, while $\mathcal{Q}_5$ itself is obtained by double coating the outside of $\mathcal{Q}$ in $\mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$. Therefore, imaginary circles are lost : when the radius decreases to 0 in an *isotomic* pencil, the differentiable continuation is going back to positive radiuses (with the other orientation) and not escaping to imaginary values.

Proposition 11.7.5. *Let Ω_0 be a cycle, but not a point-circle, σ the inversion wrt cycle Ω_0 as described in Theorem 11.4.13, $V_1 \in \mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^4\right)$ a representative of cycle Ω_1 and $Y_1 = \left(V_1, \tau_1\right) \in \mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^5\right)$ a representative of one of the corresponding oriented cycles. Then applications :*

$$\begin{array}{rcl} \sigma^+ & : & \sigma^+\left(V_1, \tau_1\right) = \left(\sigma\left(V_1\right), +\tau_1\right) \\ \sigma^- & : & \sigma^-\left(V_1, \tau_1\right) = \left(\sigma\left(V_1\right), -\tau_1\right) \end{array}$$

are describing inversions wrt each of the oriented cycles corresponding to Ω_0. The conservation law (11.15) can be rewritten in order to describe a projective invariant by inversion, namely :

$$-\frac{{}^tY_1 \boxed{\mathcal{Q}_5} Y_2}{4 S^2 \, \tau_1 \, \tau_2} = \frac{\left|P_1 P_2\right|^2 - \left(\omega_1 - \omega_2\right)^2}{2 \omega_1 \, \omega_2}$$

or, for a circle and a line,

$$-\frac{{}^{t}Y_1 \boxed{\mathcal{Q}_5} Y_3}{4S^2\,\tau_1\,\tau_3} = 1 + \frac{p_1 u_3 + q_1 v_3 + r_1 w_3}{\omega_1\,(p_1 + q_1 + r_1)\,\tau_3}$$

and for two lines :

$$-\frac{{}^{t}Y_3 \boxed{\mathcal{Q}_5} Y_4}{4S^2\,\tau_3\,\tau_4} = 1 \pm \cos\left(\Delta_3,\,\Delta_4\right)$$

Remark 11.7.6. Searby (2009) illustrates how this invariant (named ϵ_{jk}) can be used to summarize tangencies in various situations.

Chapter 12

Using complex numbers

Our aim in this chapter is to describe how to translate into complex numbers all of the ***methods*** we have described in the other chapters This is equivalent to give the complex version of all the operators we have already described. Therefore our focus will be on these operators. In fact, some of these operators have a very simple form when using complex numbers and are, for this reason threthened of misrecognizing.

12.1 Before starting

Definition 12.1.1. Complex affix of a point. Let ξ_P, η_P be the cartesian coordinates of a point P in the Euclidean plane. The $\mathbb{C}$-affix of point P is defined as

$$z_P = \xi_P + i\,\eta_P$$

Remark 12.1.2. In this definition, quantity i is a quarter turn. Since two quarter turns performed one after another is nothing but one half turn, we have $i^2 = -1$. This equation has another solution, namely $-i$, the quarter turn in the opposite orientation. Obviously, the very choice of a frame to obtain cartesian coordinates like ξ_P, η_P ensures a choice of orientation of the plane: when you look at a plane from above, you measure angles by placing your protractor onto the plane, seeing $z_P = \xi_P + i\,\eta_P$. But the guy that looks at the plane from below will put his protractor on the other face of the plane, and will therefore see $\overline{z_P} = \xi_P - i\,\eta_P$ instead of z_P.

Remark 12.1.3. The former affine point of view, describing points of $\mathbb{R}^2$ by vectors $(\xi_P,\, \eta_P,\, 1)$ can be restated as :

$$P\ :\quad \begin{pmatrix} z_P \\ 1 \\ \overline{z_P} \end{pmatrix} = \begin{pmatrix} \xi_P + i\,\eta_P \\ 1 \\ \xi_P - i\,\eta_P \end{pmatrix} = \begin{pmatrix} 1 & +i & 0 \\ 0 & 0 & 1 \\ 1 & -i & 0 \end{pmatrix} \cdot \begin{pmatrix} \xi_P \\ \eta_P \\ 1 \end{pmatrix}$$

dealing together with the from above and the from below points of view.

Theorem 12.1.4. *Given the affixes of the vertices, the* ***area of triangle*** *ABC is given by :*

$$area\,(ABC) = \left(-\frac{1}{4}\,i\right) \begin{vmatrix} z_A & z_B & z_C \\ 1 & 1 & 1 \\ \overline{z_A} & \overline{z_B} & \overline{z_C} \end{vmatrix} \tag{12.1}$$

Proof. This result has great consequences, but a very short proof. Formula (5.6) using cartesian coordinates has to be modified by factor $i/2$ since :

$$\det \begin{pmatrix} 1 & +i & 0 \\ 0 & 0 & 1 \\ 1 & -i & 0 \end{pmatrix} = 2i$$

□

Proposition 12.1.5. *If affine point P has $p:q:r$ barycentric coordinates with respect to triangle ABC, then :*

$$\begin{pmatrix} z_P \\ 1 \\ \overline{z_P} \end{pmatrix} = \frac{1}{p+q+r} \begin{pmatrix} z_A & z_B & z_C \\ 1 & 1 & 1 \\ \overline{z_A} & \overline{z_B} & \overline{z_C} \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} \tag{12.2}$$

while collinearity of three points and cocyclicity of four points are (respectively) :

$$\det_{j=1}^{3} \left([z_j,\, \overline{z_j},\, 1] \right) = 0, \quad \det_{j=1}^{4} \left([z_j\, \overline{z_j},\, z_j,\, \overline{z_j},\, 1] \right) = 0 \tag{12.3}$$

Proof. Obvious from their cartesian counterparts and the linearity of determinant. □

12.2 How to deal with complex conjugacy ?

When writing points M as $(\xi,\, \eta)$ and describing curves $\mathcal{C}$ by polynomials P so that $M \in \mathcal{C}$ when $P(\xi,\, \eta) = 0$, it is efficient to use homogeneous coordinates (X, Y, T) and describe curves by homogeneous polynomials

$$P_n(X, Y, T) = T^n\, P\left(\frac{X}{T}, \frac{Y}{T}\right)$$

where $n = dg(P)$ is the degree of the curve. Indeed, we have a theorem stating that curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have exactly $m\,n$ common points when polynomials P_n and P_m have no non-constant common factor. To obtain this result, all the obtained points have to be taken into account, including points with non real coordinates as well as points at infinity, and also considerring the multiplicities of the solutions. When we want to reformulate this result using complex affixes, we need the following proposition.

Proposition 12.2.1. *Define the complex polynomial Q_n of an algebraic curve by*

$$Q_n\left(\mathbf{Z},\, \mathbf{T},\, \overline{\mathcal{Z}}\right) = P_n\left(\frac{\mathbf{Z} + i\,\overline{\mathcal{Z}}}{2},\, \frac{\mathbf{Z} - i\,\overline{\mathcal{Z}}}{2\,i},\, \mathbf{T}\right)$$

where $\mathbf{Z}$ (read it as "big z"), $\overline{\mathcal{Z}}$ (read it as "big ζ") and $\mathbf{T}$ are algebraic variables. Define $conj\,(Q_n)$ as the complex polynomial associated to polynomial $\overline{P_n}$ (obtained by complex conjugation of the coefficients). Then

$$conj\left(\sum_{p+q+r} c_{p,\,q,\,r}\, \mathbf{Z}^p\, \overline{\mathcal{Z}}^q\, \mathbf{T}^r\right) = \sum_{p+q+r} \overline{c_{p,\,q,\,r}}\, \mathbf{Z}^q\, \overline{\mathcal{Z}}^p\, \mathbf{T}^r$$

in other words, conjugate the coefficients and exchange $\mathbf{Z}$ with $\overline{\mathcal{Z}}$.

Remark 12.2.2. An algebraical variable is a placeholder used to write polynomials. What could be the $\mathbb{C}$-conjugate of a placeholder ? Therefore, a notation was to be found in order to satisfy the following constraints :

1. Have capital letters, since polynomials are usually written $P(X_1, X_2, \cdots, X_k)$
2. Avoid indices, and deal with the fact that letter "big ζ" has the same shape as letter "big z" (nevertheless, read $\overline{\mathcal{Z}}$ as "big zeta").
3. Have a robust cursive version, to facilitate hand computations: a Z is clearly a $\mathbf{Z}$, while a $\overline{Z}$ is clearly a $\overline{\mathcal{Z}}$!
4. Enforce nevertheless the fact that $\overline{\mathcal{Z}}$ is not the $\mathbb{C}$-conjugate of $\mathbf{Z}$.

This leads to the following definition.

Definition 12.2.3. The **Morley space** is $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$: a point is a complex triple defined up to a non-zero complex factor. A point of the Morley space that can be written as :

$$\zeta_P \simeq \begin{pmatrix} z_P \\ 1 \\ \overline{z_P} \end{pmatrix}$$

for some $z_P \in \mathbb{R}^2$ is referred as the Morley-affixe of the ordinary point P (shortened as : ζ_P is a finite point). A point of the Morley space that can be written as

$$\zeta_\vartheta \simeq \begin{pmatrix} \exp(+i\,\vartheta) \\ 0 \\ \exp(-i\,\vartheta) \end{pmatrix}$$

for some $\vartheta \in \mathbb{R}$ is referred as the Morley-affixe of the angle whose measure is $\vartheta + k\,\pi$ (shortened as : ζ_ϑ is a direction). Taken together, finite points and directions are referred as the **visible point**s. All other points of the Morley space are described as being **invisible**.

Remark 12.2.4. The invisible points of the Morley space correspond to the points that cannot be written as $(x : y : t)$ with $x, y, t \in \mathbb{R}$ in the ξ, η, t representation. In this context they are referred as "non real". In the context of the Morley-space, this designation is no more suitable, and another word must be used.

Remark 12.2.5. From an abstract point of view, it could be tempting to define the Morley space as $\mathbb{P}_\mathbb{R}(\mathbb{C} \times \mathbb{C} \times \mathbb{R})$, i.e. to describe points by triples $(z,\,\zeta,\,t)$ such that $z,\,\zeta \in \mathbb{C}$ and $t \in \mathbb{R}$, each triple being defined up to a real proportionality factor. But in real life, enforcing $t \in \mathbb{R}$ can only be done by carrying annoying factors. We better have the following definition.

Proposition 12.2.6. *An algebraic curve is said to be "reduced to a point" when it contains only one visible nops(%);point, and "visible" when it contains an infinite number of visible points. The polynomial of a visible curve must be proportional to its conjugate.*

Remark 12.2.7. From an abstract point of view, using a constant factor to enforce $conj(P) = P$ is always possible. But in real life, this can only be done by carrying annoying factors and has to be avoided. Before any simplification, a polynomial obtained from a determinant, as the equation of a line or a circle, verifies $conj(P) = -P$.

Proposition 12.2.8. *In the Morley space, the equation of a visible line is* $\overline{a}\,\mathbf{Z} + a\,\overline{\mathcal{Z}} + b\mathbf{T} = 0$ *(with* $b \in \mathbb{R}$*). Its point at infinity is :*

$$a : 0 : -\overline{a} = \cos\vartheta + i\,\sin\vartheta : 0 : \cos\vartheta - i\,\sin\vartheta = \omega^2 : 0 : 1$$

where ϑ *is the oriented angle from the real axe to the line. This is an angle between straight lines and not an angle between vectors. We have the formula :*

$$\omega^2 = -\operatorname{coeff}\left(\Delta,\,\overline{\mathcal{Z}}\right) \div \operatorname{coeff}\left(\Delta, Z\right)$$

Proof. Straight line $y = p\,x + q$ can be rewritten as $(p+i)\mathbf{Z} + (p-i)\overline{\mathcal{Z}} + 2q\,\mathbf{T} = 0$. Point at infinity is $[p+i,\,2\,q,\,p-i] \wedge [0,1,0]$. ☐

Proposition 12.2.9. *In the Morley space, the equation of a visible cycle is* $p\,\mathbf{Z}\overline{\mathcal{Z}} + \left(\overline{q}\mathbf{Z} + q\overline{\mathcal{Z}}\right)\mathbf{T} + r\,\mathbf{T}^2 = 0$ *(with* p, r *in* $\mathbb{R}$*). Each circle (*$p \neq 0$ *) goes through the umbilics :*

$$\Omega_x \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \Omega_y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Proof. Circle centered at $M_0 = (z_0,\,\overline{z_0},\,1)$ and radius ω is given by :

$$(\mathbf{Z} - z_0)\left(\overline{\mathcal{Z}} - \overline{z_0}\right) - \omega^2$$

☐

Remark 12.2.10. When all of the terms are present, variable $\mathbf{T}$ can be omitted. But when substracting normalized circles, obtaining the radical axis, it makes a difference to obtain $\left(\overline{b}\mathbf{Z} + b\overline{\mathcal{Z}}\mathbf{T} + c\,\mathbf{T}\right)\mathbf{T}$ instead of a first degree polynomial !

12.3 Lubin representation of first degree

Proposition 12.3.1. *The* **Morley-affix** *of a point P whose barycentrics are $p:q:r \in \mathbb{P}_{\mathbb{R}}\left(\mathbb{R}^3\right)$ with respect to triangle ABC is given by :*

$$\zeta_P \simeq \left(\begin{array}{ccc} z_A & z_B & z_C \\ 1 & 1 & 1 \\ \overline{z_A} & \overline{z_B} & \overline{z_C} \end{array} \right) \cdot \left(\begin{array}{c} p \\ q \\ r \end{array} \right) \tag{12.4}$$

Proof. When $p+q+r \neq 0$, this is nothing but (12.2), and ζ_P is a Morley finite point. When $p+q+r=0$, P is at infinity and ζ_P is a Morley direction. The condition $p,q,r \in \mathbb{R}$ ensures that no invisible points in the Morley-space can be generated from a real point in the Kimberling space. □

Remark 12.3.2. Formula (12.4) requires a lot of conjugacies... and it is well known that conjugacy is a terrific process that introduces branching points. Therefore a method is needed to transform conjugacy into an holomorphic process, at least for a sufficiently large subset of points.

Definition 12.3.3. The Lubin parametrizations are obtained by assuming that the circumcircle of triangle ABC is nothing but the unit circle of the complex plane, together with the relations :

$$z_A = \alpha^n,\ z_B = \beta^n,\ z_C = \gamma^n$$

Since our interest is directed toward central objects, we will largely use the so-called symmetrical functions :

$$s_1 \doteq \alpha + \beta + \gamma =,\ s_2 \doteq \alpha\,\beta + \beta\,\gamma + \alpha\,\gamma,\ s_3 \doteq \alpha\,\beta\,\gamma,\ vdm = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) \tag{12.5}$$

$$\sigma_1 = z_A + z_B + z_C,\ \sigma_2 = z_A z_B + z_B z_C + z_A z_C,\ \sigma_3 = z_A z_B z_C,\ Vdm = (z_A - z_B)(z_B - z_C)(z_C - z_A)$$

Remark 12.3.4. Quantity vdm (the Vandermonde of the three numbers) is skew-symmetric and verifies :

$$vdm^2 = s_1^2 s_2^2 - 4\,s_3\,s_1^3 - 4\,s_2^3 + 18\,s_3\,s_1\,s_2 - 27\,s_3^2$$

so that vdm never appears by a power. The "small" symmetrical functions s_1, s_2, s_3, vdm depend on the degree of the representation, while the "big" ones σ_1, σ_2, σ_3, Vdm depend only on the triangle.

Proposition 12.3.5. Forward and backward matrices. *Using the Lubin-1 parametrization, we have :*

$$\boxed{Lu} = \left(\begin{array}{ccc} \alpha & \beta & \gamma \\ 1 & 1 & 1 \\ 1/\alpha & 1/\beta & 1/\gamma \end{array} \right) \quad ; \quad \det \boxed{Lu} = \frac{Vdm}{\sigma_3} = \frac{-4i}{R^2}\,S \tag{12.6}$$

$$\boxed{Lu^{-1}} = \frac{1}{Vdm} \left[\begin{array}{ccc} \alpha\,(\beta-\gamma) & \alpha\left(\gamma^2-\beta^2\right) & \alpha\,\beta\,\gamma\,(\beta-\gamma) \\ \beta\,(\gamma-\alpha) & \beta\left(\alpha^2-\gamma^2\right) & \alpha\,\beta\,\gamma\,(\gamma-\alpha) \\ \gamma\,(\alpha-\beta) & \gamma\left(\beta^2-\alpha^2\right) & \alpha\,\beta\,\gamma\,(\alpha-\beta) \end{array} \right]$$

Theorem 12.3.6. Forward substitutions. *Suppose that barycentrics $p:q:r$ of point P depends rationally on a^2, b^2, c^2, S. Then Morley-affix of P is obtained by substituting the identities :*

$$S = \frac{i\,R^2\,(\alpha-\beta)(\gamma-\alpha)(\beta-\gamma)}{4\,\alpha\,\beta\,\gamma},\ a = \frac{R\,(\beta-\gamma)}{\sqrt{-\beta\,\gamma}},\ b = \frac{R\,(\gamma-\alpha)}{\sqrt{-\gamma\,\alpha}},\ c = \frac{R\,(\alpha-\beta)}{\sqrt{-\alpha\,\beta}} \tag{12.7}$$

into $p:q:r$ and applying (12.4). The result obtained is a rational fraction in α, β, γ whose degree is $+1$. When P is a triangle center, ζ_P depends only on σ_1, σ_2, σ_3. When P is invariant by circular permutation, but not by transposition, a Vdm term appears.

Proof. Cancellation of radicals is assured by condition $\mathbb{Q}\left(a^2, b^2, c^2, S\right)$. Elimination of R comes from homogeneity. Symmetry properties are evident. □

Proposition 12.3.7. *In the Morley space,* **line at infinity** *is described by*

$$\boxed{\begin{matrix} 0 & 1 & 0 \end{matrix}} = \mathcal{L}_\infty \,.\, \boxed{Lu^{-1}} \tag{12.8}$$

Proof. The value is obvious, while the formula describes what happens to linear forms during a change of basis. □

Proposition 12.3.8. *In the Morley space, the matrix* $\boxed{W_z}$ *of the operator that changes the affix* D *of a line, into the affix* P *of the point at infinity of that line can be written as :*

$$P = \boxed{W_z} \,.\, {}^t D \qquad \text{where } \boxed{W_z} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix} \simeq \boxed{Lu} \,.\, \boxed{W} \,.\, {}^t\boxed{Lu} \tag{12.9}$$

Proof. Change of basis. The result can also be deduced from the fact that operator $\wedge$ is the universal factorization of the determinant. This explains why the proportionality factor (acting over the right member of the $\simeq$) is exactly $1/\det \boxed{Lu}$. □

Proposition 12.3.9. *In the Morley space, the matrix* $\boxed{Pyth_z}$ *of the quadratic form that give the squared length of a vector in the* $\mathcal{V}$ *space can be written as :*

$$\boxed{Pyth_z} = R^2\, \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \boxed{Lu} \,.\, \boxed{W} \,.\, {}^t\boxed{Lu} \div \left(\det \boxed{Lu}\right)^2 \tag{12.10}$$

Proof. Here, equalities are required since a length is not defined up to a proportionality factor. Remember that an element of $\mathcal{V}$ is obtained as the difference of the normalized columns of two finite points, and therefore looks like $z_1/t_1 - z_2/t_2,\ 0,\ \overline{z_1}/t_1 - \overline{z_2}/t_2$. The obtained formula is nothing but the usual $|\zeta|^2 = \zeta\,\overline{\zeta}$. The strange looking -2 that acts on the $t = 0$ component is the counterpart for the better loking form of the $\boxed{Pyth}$ matrix. □

Proposition 12.3.10. *In the Morley space, the matrix* $\boxed{OrtO_z}$ *of the operator that transforms a direction into its orthogonal direction while transforming the orthocenter into* $0:0:0$*, is given by :*

$$\boxed{OrtO_z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \simeq \boxed{Lu} \,.\, \boxed{OrtO} \,.\, \boxed{Lu^{-1}} \tag{12.11}$$

Proof. Expression of $\boxed{OrtO}$ using a, b, c is given (5.16). Then substitutions (12.7) are to be used. Eigenvectors of $\boxed{OrtO_z}$ are both umbilics $1:0:0$, $0:0:1$ and $0:1:0$ (the circumcenter). □

Proposition 12.3.11. Orthodir. *In the Morley space, the matrix* $\boxed{\mathcal{M}_z}$ *of the orthodir operator that changes the affix* D *of a line, into the affix* P *of the point at infinity in the orthogonal direction, can be written as :*

$$P = \boxed{\mathcal{M}_z} \,.\, {}^t D \qquad \text{where } \boxed{\mathcal{M}_z} \doteq i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = i \boxed{OrtO_z} \,.\, \boxed{W_z} \simeq \boxed{Lu} \,.\, \boxed{\mathcal{M}} \,.\, {}^t\boxed{Lu} \tag{12.12}$$

Proof. Obvious from definitions. Coefficient i disappeared when defining matrix $\boxed{OrtO_z}$, and is reappearing here in order to obtain a better looking tangent formula. □

Theorem 12.3.12. Tangent of two lines. *In the Morley space, the oriented angle from a visible line* Δ_1 *to a visible line* Δ_2 *is characterized by :*

$$\tan\left(\widehat{\Delta_1, \Delta_2}\right) = i\, \frac{p_2\, q_1 - p_1\, q_2}{p_2\, q_1 + p_1\, q_2} = \frac{\Delta_1 \,.\, \boxed{W_z} \,.\, {}^t\Delta_2}{\Delta_1 \,.\, \boxed{\mathcal{M}_z} \,.\, {}^t\Delta_2} \tag{12.13}$$

where $\boxed{W_z}$, $\boxed{\mathcal{M}_z}$ *are exactly as given in* (12.9) *and* (12.12) *(not up to a proportionality factor).*

Proof. Can be obtained from elementary methods. It is nevertheless interresting to reobtain these results from former results. Numerator tell us when lines are parallel, and denominator when they are orthogonal. □

Proposition 12.3.13. Backward substitutions. *Let $z_P \in \mathbb{C}(\alpha,\, \beta,\, \gamma)$ be an homogeneous rational fraction, supposed to be the complex affix of the finite point. Then* $\deg(z_P) = 1$ *is required. Alternatively, let $\omega^2 \in \mathbb{C}(\alpha,\, \beta,\, \gamma)$ is an homogeneous rational fraction, supposed to describe the Morley affix of a direction. Then* $\deg\left(\omega^2\right) = 2$ *is required. When these conditions are fulfilled, the ABC-barycentrics $p : q : r$ of these objects can be obtained as follows. Compute the corresponding vector :*

$$\left(\begin{array}{c} p \\ q \\ r \end{array}\right) = \boxed{Lu}^{-1} \cdot \left(\begin{array}{c} z_P \\ 1 \\ \overline{z_P} \end{array}\right) \quad \text{ou} \quad \left(\begin{array}{c} p \\ q \\ r \end{array}\right) = \boxed{Lu}^{-1} \cdot \left(\begin{array}{c} \omega^2 \\ 0 \\ 1 \end{array}\right)$$

then apply substitutions

$$\begin{aligned} \beta &= \left(\frac{+2\,i\left(a^2+b^2-c^2\right)}{a^2b^2}\,S + \frac{a^4+b^4+c^4-2\left(a^2+b^2\right)c^2}{2\,a^2b^2}\right)\alpha \\ \gamma &= \left(\frac{-2\,i\left(c^2+a^2-b^2\right)}{a^2c^2}\,S + \frac{a^4+b^4+c^4-2\left(a^2+c^2\right)b^2}{2\,a^2c^2}\right)\alpha \end{aligned}$$

to this vector and simplify the obtained expression using the Heron formula :

$$S^2 = -\frac{1}{16}\,(a+b+c)\,(b+c-a)\,(c+a-b)\,(a+b-c)$$

Proof. Transform $\alpha,\, \beta,\, \gamma$ into $\alpha\delta,\, \beta\delta,\, \gamma\delta$. Since this transform is a similitude, barycentrics must remain unchanged and the z_P affix is turned by δ. On the other hand, polynomial z_P is homogeneous and z_P is multiplied by δ^k where $k = \deg(z_P)$. Concerning the directions, $\arg\left(\omega^2\right)$ is twice the angle with the real axis, and degree 2 is required.

Alternatively, the degrees of rows of ζ_P are $+k,\, -k,\, 0$ while the degrees of columns of $\boxed{Lu}^{-1}$ are $-1,\, 0,\, +1$. Quantities p, q, r will therefore be a sum of terms whose degrees are respectively $k-1,\, 0,\, 1-k$. But homogeneity is required in order that a transformation $\beta = B\,\alpha$, $\gamma = C\,\alpha$ can eliminate α, leading to $k = 1$.

To obtain the substitution formulae, compute β from $c^2\,\alpha\,\beta = -R^2\,(\alpha-\beta)^2$. A choice of branch (a sign for i) has to be chosen. Exchange b and c (and therefore change S into $-S$) and obtain the corresponding γ. □

Remark 12.3.14. When starting with a symmetrical Morley-affix, the obtained $p : q : r$ remains symmetrical in $\alpha,\, \beta,\, \gamma$. The given substitutions are breaching the symmetry of individual coefficients p, q, r, that can only be reestablished by cancellation of unsymmetrical common factors between the p, q, r. Most of the time, its more efficient to proceed by numerical substitution and use the obtained search key to identify the point (and proceed back to obtain a proof of the result).

Proposition 12.3.15. *The Kimberling search key associated to a visible finite point defined by its Morley affix (short= Morley's search key) is obtained by substituting :*

$$\alpha = 1\,;\; \beta = -\frac{391}{729} - i\,\frac{104}{729}\,\sqrt{35}\,;\; \gamma = \frac{401}{1521} - i\,\frac{248}{1521}\,\sqrt{35}$$

into $\mathbf{Z}/\mathbf{T}$ *and then computing :*

$$searchkey\left(\frac{\mathbf{Z}}{\mathbf{T}}\right) \doteq \Re\left(\left(\frac{157}{840}\,\sqrt{35} - i\,\frac{22}{3}\right)\frac{\mathbf{Z}}{\mathbf{T}}\right) + \frac{321}{280}\,\sqrt{35}$$

Proof. Kimberling's search keys are associated with triangle $a = 6$, $b = 9$, $c = 13$. The radius of the circumcircle is $R = (351/280)\sqrt{35}$. One can see that sidelengths of triangle $\alpha\beta\gamma$ are $6/R$, $9/R$, $13/R$. We apply these substitutions to obtain the numerical value of the "retour" matrix, and then use (1.3) □

Proposition 12.3.16. *The Morley's searchkey of a visible point at infinity ($\mathbf{T}=0$) is obtained from $\Omega=\mathbf{Z}/\overline{\mathcal{Z}}$ by :*

$$\frac{1108809\left(241+16\,i\sqrt{35}\right)\Omega^{2}-907686\left(157+176\,i\sqrt{35}\right)\Omega+\left(-224394311+30270800\,i\sqrt{35}\right)}{389191959\,\Omega^{2}-\left(106136082+118980576\,i\sqrt{35}\right)\Omega+\left(19397664\,i\sqrt{35}-371888361\right)}$$

Another method is identifying the isogonal conjugate of the given point, which is simply : σ_3/Ω : $-1:\Omega/\sigma_3$.

Proof. The searchkey of an point at infinity is : $\frac{x}{a}\times\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)$, leading to this tremendous expression. But after all, this formula is not designed for hand computation but rather to a floating evaluation by a computer.... . □

12.4 Some examples of first degree

Example 12.4.1. The circumcenter O. By definition, $\zeta_O=0:0:1$. The preceeding transformations are giving :

$$\left(\begin{array}{c}p\\q\\r\end{array}\right)\simeq\left(\begin{array}{c}\alpha\left(\beta-\gamma\right)\left(\beta+\gamma\right)\\\beta\left(\gamma-\alpha\right)\left(\alpha+\gamma\right)\\\gamma\left(\alpha-\beta\right)\left(\alpha+\beta\right)\end{array}\right)\simeq\left(\begin{array}{c}a^{2}\left(b^{2}+c^{2}-a^{2}\right)\\b^{2}\left(c^{2}+a^{2}-b^{2}\right)\\c^{2}\left(a^{2}+b^{2}-c^{2}\right)\end{array}\right)$$

Example 12.4.2. Symmedian point X(6), aka Lemoine point.

1. Consider the middle A' of segment $[B,C]$ and define the A symmedian as the line Δ_A that goes through A and verifies $\angle(AB,\Delta_A)=\angle(AA',AC)$. We will use P instead ζ_P since this is more readable ... and hand-writable. A remark : symmetry wrt bissectors is irrelevant, since bissectors are unrecheable !

2. We have $A'=B+C$, $AA'=A\wedge A'$ etc. and our equations are

$$\begin{array}{rcl}\Delta_A\,.\,A&=&0\\\tan\left(AB,\,\Delta_A\right)+\tan\left(AC,\,AA'\right)&=&0\end{array}$$

   solving this system, then permutating, gives :

$$\begin{array}{rcl}\Delta_A&\simeq&\left(\begin{array}{ccccc}2\,\alpha-\beta-\gamma&;&\left(\alpha\,\gamma+\alpha\,\beta-2\,\beta\,\gamma\right)\alpha&;&-2\,\alpha^{2}+2\,\beta\,\gamma\end{array}\right)\\\Delta_B&\simeq&\left(\begin{array}{ccccc}2\,\beta-\gamma-\alpha&;&\left(\alpha\,\beta+\beta\,\gamma-2\,\alpha\,\gamma\right)\beta&;&-2\,\beta^{2}+2\,\alpha\,\gamma\end{array}\right)\end{array}$$

3. Intersecting two symmedians gives a symmetrical result. Therefore, the three symmedians are concurrent at some point. This point is well known as Lemoine point, and we have :

$$K=\zeta\left(6\right)=\Delta_A\wedge\Delta_B=\left(\begin{array}{c}2\,\sigma_2^{2}-6\,\sigma_3\,\sigma_1\\\sigma_2\,\sigma_1-9\,\sigma_3\\2\,\sigma_1^{2}-6\,\sigma_2\end{array}\right)$$

4. Going back to barycentrics, we obtain the well known result :

$$X\left(6\right)\simeq\left(\begin{array}{c}\alpha\left(\gamma-\beta\right)^{2}\\\beta\left(\alpha-\gamma\right)^{2}\\\gamma\left(\alpha-\beta\right)^{2}\end{array}\right)\simeq\left(\begin{array}{c}a^{2}\\b^{2}\\c^{2}\end{array}\right)$$

Example 12.4.3. The Kiepert parabola.

1. Morley equation of the circumcircle is $\mathbf{Z}\,\overline{\mathcal{Z}}-\mathbf{T}^{2}=0$. The Morley affix Δ_P of the polar line of point $K=z:1:\zeta$ wrt the circumcircle is therefore given by :

$$\Gamma_P\doteq\left[\begin{array}{ccc}z&1&\zeta\end{array}\right].\left(\begin{array}{ccc}0&0&1\\0&-2&0\\1&0&0\end{array}\right)$$

2. The coefficients of the tangential conic determined by five given lines $[u_j, v_j, w_j]$ are obtained as :
$$\bigwedge_{j=1..5} \left[u_j^2,\, v_j^2,\, w_j^2,\, u_j v_j,\, v_j w_j,\, w_j u_j\right]$$
by universal factorization of the corresponding 6×6 determinant. Let us consider the inconic tangent to the infinity line (parabola) and to the circular polar of point K. Using $(BC) \simeq B \wedge C$, etc together with the previous equation, we obtain the symmetric matrix :
$$\boxed{\mathcal{C}^*} \simeq \begin{pmatrix} 2\,\sigma_3\, z^2 - 2\,\sigma_2\,\sigma_3\, z\,\zeta + 4\,\sigma_3^2\,\zeta & qsp & -\sigma_1\, z^2 + \sigma_2\,\sigma_3\,\zeta^2 + 2\,\sigma_2\, z - 2\,\sigma_3\,\sigma_1\,\zeta \\ -\sigma_3\,\sigma_1\, z\zeta + \sigma_3^2\zeta^2 + 2\,\sigma_3\, z & 0 & -z^2 + \sigma_2\, z\,\zeta - 2\,\sigma_3\,\zeta \\ -\sigma_1\, z^2 + \sigma_2\,\sigma_3\,\zeta^2 + 2\,\sigma_2\, z - 2\,\sigma_3\,\sigma_1\,\zeta & qsp & 2\,\sigma_1\, z\,\zeta - 2\,\sigma_3\,\zeta^2 - 4\, z \end{pmatrix}$$
Degrees of all these expressions are :
$$\mathrm{dg}\left(\boxed{\mathcal{C}^*}\right) = \begin{pmatrix} 5 & 4 & 3 \\ 4 & . & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

3. A focus is a point such that both isotropic lines throught that point are tangent to the conic. Writing that $Q = \left(\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}\right) \wedge (1:0:0)$ satisfies $Q\,.\,\boxed{\mathcal{C}^*}\,.\,{}^tQ = 0$ and the similar with the other umbilic gives two equations whose solution is
$$F\,(z) \simeq \begin{pmatrix} \dfrac{z^2 - \sigma_2\, z\zeta + 2\,\sigma_3\,\zeta}{2\,z - \sigma_1\, z\zeta + \sigma_3\,\zeta^2} \\ 1 \\ \dfrac{2\,z - \sigma_1\, z\zeta + \sigma_3\,\zeta^2}{z^2 - \sigma_2\, z\zeta + 2\,\sigma_3\,\zeta} \end{pmatrix}$$
and this point is on the circumcircle.

4. Take K at the Lemoine point X(6). Its circumpolar line is called the Lemoine axis. One obtains the Kiepert parabola, where :
$$\boxed{\mathcal{C}^*} \simeq \begin{pmatrix} 2\,\sigma_2\,\sigma_3 & \sigma_3\,\sigma_1 & 0 \\ \sigma_3\,\sigma_1 & 0 & -\sigma_2 \\ 0 & -\sigma_2 & -2\,\sigma_1 \end{pmatrix},\ \boxed{\mathcal{C}} \simeq \begin{pmatrix} -\sigma_2^2 & 2\,\sigma_3\,\sigma_1^2 & -\sigma_2\,\sigma_3\,\sigma_1 \\ 2\,\sigma_3\,\sigma_1^2 & -4\,\sigma_2\,\sigma_3\,\sigma_1 & 2\,\sigma_2^2\,\sigma_3 \\ -\sigma_2\,\sigma_3\,\sigma_1 & 2\,\sigma_2^2\sigma_3 & -\sigma_1^2\,\sigma_3^2 \end{pmatrix}$$

5. As stated in Proposition 9.3.14, the triangle of the circle-polars of the sidelines of triangle $\boxed{\mathcal{T}}$ is described by matrix $\boxed{\mathcal{C}^*}\,.\,{}^t\boxed{\mathcal{T}^*}$. Both triangle are in perspective (lines AA', etc are concurrent). The perspector is obtained as $AA' \wedge BB'$ and concurrence is verified by the symmetry of the result. It is well known that $P =$X(99), the Steiner point.
$$F = \frac{\sigma_2}{\sigma_1},\ P = \frac{\sigma_3\sigma_1^2 - 3\,\sigma_2\sigma_3}{\sigma_2^2 - 3\,\sigma_3\sigma_1}$$

6. Applying the preceeding transformations, the $p:q:r$ associated with $z = \sigma_2/\sigma_1$ is obtained as :
$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \simeq \begin{pmatrix} \alpha\,(\beta-\gamma)\left(\alpha\,\beta-\gamma^2\right)\left(\gamma\,\alpha-\beta^2\right) \\ \beta\,(\gamma-\alpha)\left(\beta\,\gamma-\alpha^2\right)\left(\alpha\,\beta-\gamma^2\right) \\ \gamma\,(\alpha-\beta)\left(\gamma\,\alpha-\beta^2\right)\left(\beta\,\gamma-\alpha^2\right) \end{pmatrix} \simeq \begin{pmatrix} \dfrac{a^2}{(b+c)\,(b-c)} \\ \dfrac{b^2}{(c+a)\,(c-a)} \\ \dfrac{c^2}{(a+b)\,(a-b)} \end{pmatrix}$$
and we can identify X(110), the focus of the Kiepert parabola.

Example 12.4.4. Isogonic, isodynamic and Napoleon.

1. Define $j = \exp(2\,i\pi/3)$. Start from triangle ABC. Construct P_A such triangle P_ABC is equilateral. More precisely, the z affix of P_A is such that $z + j\,\beta + j^2\,\gamma = 0$, deciding of the orientation. The three lines AP_A, BP_B, CP_C are concurrent leading to the first isogonic center. Changing j into j^2 leads to the second isogonic point. A simple computation leads to :
$$\frac{9\,\sigma_2\sigma_3 - 12\,\sigma_3\sigma_1^2 + 3\,\sigma_1\sigma_2^2}{6\,\sigma_2^2 - 18\,\sigma_3\sigma_1} \pm i\sqrt{3}\frac{vdm\,\sigma_2}{6\,\sigma_2^2 - 18\,\sigma_3\sigma_1}$$

2. When trying to transform the former expression into barycentrics, the formal computer is poisonned by the folowing fact. Quantity vdm describes the orientation of the triangle, while $\sqrt{3}$ describes the orientation of the whole plane. We better generalize the problem using $\tan(AB,\,AA') = K$. This leads to :
$$\zeta_K \simeq \begin{pmatrix} 4\,K\left(\sigma_2^2 - 3\,\sigma_3\,\sigma_1\right) + i\,vdm\left(K^2+1\right)\sigma_1 \\ 2\,K\left(\sigma_2\,\sigma_1 - 9\,\sigma_3\right) + i\,vdm\left(3+K^2\right) \\ 4\,K\left(\sigma_1^2 - 3\,\sigma_2\right) + i\,vdm\left(K^2+1\right)\dfrac{\sigma_2}{\sigma_3} \end{pmatrix}$$
$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \simeq \begin{pmatrix} \dfrac{\gamma-\beta}{(\beta+\gamma)\,K + i\,(\gamma-\beta)} \\ \dfrac{\alpha-\gamma}{(\alpha+\gamma)\,K + i\,(\alpha-\gamma)} \\ \dfrac{\beta-\alpha}{(\alpha+\beta)\,K + i\,(\beta-\alpha)} \end{pmatrix} \simeq \begin{pmatrix} \dfrac{1}{2\,S - S_aK} \\ \dfrac{1}{2\,S - S_bK} \\ \dfrac{1}{2\,S - S_cK} \end{pmatrix}$$

3. And we obtain a lot of results when changing K, and even more by isogonal conjugacy. In the following table, line K lists usual values for the tangent of an angle, while the other two lines give the Kimberling number of the corresponding points. The P_K points are on the Kiepert RH (more details in Proposition 10.18.1).

K	$-\sqrt{3}$	-1	$\frac{-1}{2}$	$\frac{-1}{\sqrt{3}}$	0	$\frac{+1}{\sqrt{3}}$	$\frac{+1}{2}$	1	$\sqrt{3}$	∞
P_K	13	485	3316	17	2	18	3317	486	14	4
$isog\,(P_K)$	15	371	3311	61	6	62	3312	372	16	3

Fact 12.4.5. *As of jmax* < 3587, *there are 367 points whose barycentrics contain R. Among them :*

1. 49 contain other litteral radicandes
2. 85 are are in $\mathbb{C}(a,\,b,\,c,\,S)$ but aren't in $\mathbb{C}\left(a^2,\,b^2,\,c^2,\,S\right)$
3. 233 are in fact inside $\mathbb{C}\left(a^2,\,b^2,\,c^2,\,S\right)$. When changing R into $-R$, three arn't paired 2981, 3381, 3382: there are 115 pairs. For 46 of them, the isogonal of a known pair is again a known pair.

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Chapter 13

Collineations

13.1 Definition

Definition 13.1.1. A collineation is a reversible linear transformation of the barycentrics, i.e. $U = \phi(P)$ determined by :

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \simeq \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

where $\simeq$ is a reminder of the fact that barycentrics are determined up a proportionality factor.

Proposition 13.1.2. *A collineation is determined by two ordered lists four points: $P_i,\ i = 1, 2, 3, 4$, U_i, $i = 1, 2, 3, 4$ such that no triples of P points are on the same line, and the same for the Q points.*

Proof. If ϕ is reversible, then $\det M \neq 0$ is required and the $\phi(P_i)$ haven't alignments when the P_i haven't. Conversely, we have the following algorithm. □

Algorithm 13.1.3. Collineation algorithm. Let be given the two lists of points $P_i,\ i = 1, 2, 3, 4$, U_i, $i = 1, 2, 3, 4$. With obvious notations, the question is to find the m_{ij} (not all being 0) and the k_i (none being 0) in order to ensure :

$$\begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix} \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ r_1 & r_2 & r_3 & r_4 \end{pmatrix}$$

This system has 13 unknowns and 12 equations, since a global proportionality factor remains undetermined. The k_i are determined (up to a global proportionality factor) by

$$\frac{1}{k_i} \det \underset{\neq i}{Q} = \det \underset{\neq i}{P} \det M$$

where a 3×4 matrix subscribed by an $\neq i$ refer to the square matrice obtained by deleting the i-th column. Thereafter, M is easy to obtain. To summarize :

$$\boxed{k_i = \det \underset{\neq i}{Q} \div \det \underset{\neq i}{P} \qquad M = \underset{\neq 4}{Q} \, . \underset{\neq 4}{K} \, . \left(\underset{\neq 4}{P} \right)^{-1}}$$

With the given hypotheses, transformation ϕ is clearly reversible.

Remark 13.1.4. An efficient choice of the P_i, Q_i is eight centers, or a central triangle and a center for the P and the corresponding Q. In such a case, any center is transformed into a center, and homogenous curves into homogeneous curves of the same degree.

13.2 Involutive collineations

Proposition 13.2.1. . *Let M_1, M_2, N_1, N_2 be four (different points). The collineation ψ that swaps the $(M_1,\, M_2)$ pair and also the $(N_1,\, N_2)$ pair is involutive. The line through the crossed intersections $M_1N_1 \cap M_2N_2$ and $M_1N_2 \cap M_2N_1$ is a line of fixed points (the axe of ψ). The paired intersection, i.e. point $P = M_1M_2 \cap N_1N_2$ is an isolated fixed point (the pole of ψ). Reciprocally, given an axe Δ and a pole P (outside of the axe), we obtain an involutive transform $U \mapsto X = \psi(U)$ by requiring P, U, X aligned together with $(P,\, U,\, PU \cap \Delta,\, X) = -1$ (harmonic conjugacy).*

Proof. Consider ψ defined by $A \longleftrightarrow P$ and $B \longleftrightarrow C$. Its matrix $\boxed{\psi}$ can be obtained by the general 13.1.3 . Then the cevian triangle of P provides a diagonalization basis and we have :

$$\boxed{\psi} \simeq \left(\begin{array}{ccc} 1 & 0 & 0 \\ \dfrac{q}{p} & 0 & -\dfrac{q}{r} \\ \dfrac{r}{p} & -\dfrac{r}{q} & 0 \end{array}\right) \quad ; \quad \boxed{\mathcal{T}_P} = \left(\begin{array}{ccc} 0 & p & p \\ q & 0 & q \\ r & r & 0 \end{array}\right)$$

$$\boxed{\mathcal{T}_P}^{-1} . \boxed{\psi} . \boxed{\mathcal{T}_P} = \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

In the general case, we can chose matrix $\boxed{\psi}$ to enforce $\det \boxed{\psi} = -1$. Then minimal polynomial is $\mu^2 - 1$ while characteristic polynomial is $(\mu - 1)^2 (\mu + 1)$. □

13.3 Usual affine transforms as collineations

Remark 13.3.1. Umbilics have been defined in Section 11.2. A possible choice can be described as $\Omega^{\pm} \simeq abc\, X_{512} \pm iR\, X_{511}$. The exact value is given in (11.6).

Proposition 13.3.2. ***Translation***. *The matrix of the translation $U \mapsto U + \overrightarrow{V}$ where $\overrightarrow{V} = (p, q, r)$ is given by :*

$$\left(\begin{array}{ccc} 1+p & p & p \\ q & 1+q & q \\ r & r & 1+r \end{array}\right) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) + \overrightarrow{V} . \mathcal{L}_\infty \tag{13.1}$$

Proof. Use $P_i = A,\, B,\, \Omega^+,\, \Omega^-$ and $Q_i = C,\, D,\, \Omega^+,\, \Omega^-$. Characteristic polynomial is

$$(X-1)^2 (X - 1 - p - q - r)$$

For a translation, $p + q + r = 0$, and the matrix is not diagonalizable. □

Remark 13.3.3. The translation operator is linear, meaning that $M\left(\overrightarrow{V_1}\right) + M\left(\overrightarrow{V_2}\right) = M\left(\overrightarrow{V_1} + \overrightarrow{V_2}\right)$.

Proposition 13.3.4. ***Homothecy***. *When $p + q + r$ is different from 0 and -1 then (13.1) characterizes the homothecy centered at point $P = p : q : r$ with ratio $\mu = 1/\left(1 + p + q + r\right)$.*

Proof. The factor is the reciprocal of the eigenvalue λ_P since a fixed point should be described by $\lambda = 1$: all the eigenvalues have to be divided by λ_P. This result can also be obtained by direct examination of $\overrightarrow{f(P)\, f(U)}$. □

Remark 13.3.5. When computing P, the column ${}^t(p, q, r)$ can be viewed as "defined up to a proportionality". This does not apply to the computation of μ. In any case, we are re-obtaining (5.24).

Proposition 13.3.6. ***Similitude***. *When A, B, C, D are points at finite distance, with $A \neq B$, $C \neq D$ it exists two similitudes ϕ, ψ, one forwards and the other backwards, that sends $A \mapsto C$ and $B \mapsto D$. As collineations, they are characterized by :*

$$\begin{aligned} \phi &= \operatorname{collineate}\left(A,\, B,\, \Omega^+,\, \Omega^- ;\, C,\, D,\, \Omega^+,\, \Omega^-\right) \\ \psi &= \operatorname{collineate}\left(A,\, B,\, \Omega^+,\, \Omega^- ;\, C,\, D,\, \Omega^-,\, \Omega^+\right) \end{aligned}$$

Proof. The group of all the similitudes is the stabilizer subgroup of the pair $\{\Omega^+, \Omega^-\}$ under the action of the group of all the collineations. This comes from the fact that any similitude transforms circles into circles, and therefore must preserve the umbilical pair. $\square$

Remark 13.3.7. The matrix π_Δ of the orthogonal projector onto line $\Delta \simeq [p, q, r]$ is :

$$\pi_\Delta = \Delta \,.\, \boxed{\mathcal{M}} \,.{}^t\Delta - \boxed{\mathcal{M}} \,.{}^t\Delta \,.\, \Delta$$

while the matrix σ_Δ of the orthogonal reflection wrt line $\Delta \simeq [p, q, r]$ is :

$$\sigma_\Delta = \Delta \,.\, \boxed{\mathcal{M}} \,.{}^t\Delta - 2\boxed{\mathcal{M}} \,.{}^t\Delta \,.\, \Delta$$

These formulae are recalled from Section 5.8, where more details are given.

Proposition 13.3.8. *The matrix of the rotation centered at finite point $P = p : q : r$ with angle ϕ is :*

$$\boxed{\Phi} = \begin{pmatrix} p & p & p \\ q & q & q \\ r & r & r \end{pmatrix} + \begin{pmatrix} r+q & -p & -p \\ -q & r+p & -q \\ -r & -r & q+p \end{pmatrix} \cos\phi + \boxed{OrtO} (\mathcal{L}_\infty \,.\, P - P.\mathcal{L}_\infty) \sin\phi$$

Proof. It suffices to check what happens to P, Ω^+, Ω^- : they are fixed points, with respective eigenvalues : 1, $\exp(+i\phi)$, $\exp(-i\phi)$ — up to a global factor $p + q + r$. $\square$

Stratospherical proof. A rotation with angle ϕ is multiplication by $\Phi = \cos\phi + i \sin\phi$ in the complex plane. Therefore, rotation with center P and angle ϕ can be written as :

$$\Phi(X) = P + \left(\boxed{1}\cos\phi + \boxed{i}\sin\phi\right) \overrightarrow{PX}$$

Since matrix $\boxed{OrtO}$ describes a "project and turn" action, we have $\boxed{OrtO}^3 = -\boxed{OrtO}$, so that $\boxed{i} = \boxed{OrtO}$. Multiplying, we get : $\boxed{OrtO}^4 = -\boxed{OrtO}^2$ and $-\boxed{OrtO}^2$ is a projector onto space $\mathcal{V}$. This gives : $\boxed{1} = -\boxed{OrtO}^2$. Canceling the denominators, we obtain :

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} \simeq (x + y + z) \begin{pmatrix} p \\ q \\ r \end{pmatrix} +$$

$$\left(\sin\phi \boxed{OrtO} - \cos\phi \boxed{OrtO}^2\right) \left((p + q + r) \begin{pmatrix} x \\ y \\ z \end{pmatrix} - (x + y + z) \begin{pmatrix} p \\ q \\ r \end{pmatrix}\right)$$

leading to the required matrix :

$$\boxed{\Phi} = (\mathcal{L}_\infty \,.\, P)\; Q + (1 - Q)\;.\, P \,.\, \mathcal{L}_\infty \quad \text{where} \quad Q \doteq \left(\sin\phi \boxed{OrtO} - \cos\phi \boxed{OrtO}^2\right)$$ $\square$

13.4 Barycentric multiplication as a collineation

Proposition 13.4.1. *Barycentric multiplication by $P = p : q : r$ is what happens to the plane when using collineation $\phi : (A, B, C, X_2) \mapsto (A, B, C, P)$. In other words :*

$$X *_b P = \boxed{P} \,.\, X \qquad \text{where} \boxed{P} \doteq \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix}$$

Remark 13.4.2. Obviously, trilinear multiplication can be described using collineations involving X_1.

Proposition 13.4.3. *The collineation whose matrix is diagonal, with elements $U \div_b P$ transforms (A, B, C, P) into (A, B, C, U) and circumconic $CC(P)$ into $CC(U)$.*

Proof. Direct examination. One obtains :

$$\frac{pqr}{uvw} {}^t\begin{pmatrix} \frac{u}{p} & 0 & 0 \\ 0 & \frac{v}{q} & 0 \\ 0 & 0 & \frac{w}{r} \end{pmatrix} \cdot \begin{pmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{u}{p} & 0 & 0 \\ 0 & \frac{v}{q} & 0 \\ 0 & 0 & \frac{w}{r} \end{pmatrix} = \begin{pmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{pmatrix}$$

□

13.5 Complement and anticomplement as collineations

Proposition 13.5.1. *Complement is what happens to the plane when using collineation $(A, B, C, X_2) \mapsto (A_2B_2C_2, X_2)$ where $A_2B_2C_2$ is the medial triangle. In other words :*

$$complem\,(X) = \boxed{\mathcal{C}}\,.\,X \qquad \text{where} \quad \boxed{\mathcal{C}} \doteq \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$anticomplem\,(X) = \boxed{\mathcal{C}^{-1}}\,.\,X \qquad \text{where} \quad \boxed{\mathcal{C}^{-1}} \doteq \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Proof. Direct computation. □

Proposition 13.5.2. *The* ***cevian collineation*** *wrt point P is defined as collineation $(A, B, C, P) \mapsto (A_PB_PC_P, P)$. Its matrix is :*

$$\phi_P = \boxed{P}\,.\,\boxed{\mathcal{C}}\,.\,\boxed{P^{-1}} = \begin{pmatrix} 0 & p/q & p/r \\ q/p & 0 & q/r \\ r/p & r/q & 0 \end{pmatrix}$$

where $\boxed{P^{-1}}$ is to be understood as the reciprocal of matrix $\boxed{P}$.

Proof. Composition of the two former collineations. □

Proposition 13.5.3. *We have the following relation between conics :*

$$complem\,(X) \in conicev\,(isotom\,(U)\,,\, X_2) \iff X \in conicir\,(complem\,(U))$$

13.6 Collineations and cevamul, cevadiv, crossmul, crossdiv

In this Section 13.6, the start point will ever be Table 3.2 (II) i.e. $\mathcal{T}_1 = \mathcal{C}_P$ (the cevian of P), $\mathcal{T}_2 = ABC$, $\mathcal{T}_3 = \mathcal{A}_U$ (the anticevian of U).

Proposition 13.6.1. *Start as described, and use $\phi_U = \boxed{U}\,.\,\boxed{\mathcal{C}}\,.\,\boxed{U^{-1}}$. This collineation is tailored so that : $\phi(\mathcal{T}_3) = ABC$, $\phi(\mathcal{T}_2) = \mathcal{C}_U$ and $\phi(\mathcal{T}_1)$ is the cevian triangle of $\phi(P)$ wrt $\mathcal{C}_U$. Then :*

$$\begin{array}{rcccl} \phi\,.\,cevadiv\,(P,\,U) & = & crossdiv\,(\phi\,.\,P,\,\phi\,.\,U) & = & \frac{u^2}{p} : \frac{v^2}{q} : \frac{w^2}{r} = P_U^{\#} \\ cevamul\left(\phi^{-1}.X,\,\phi^{-1}.U\right) & = & \phi^{-1}\,.\,crossmul\,(X,\,U) & = & \frac{u^2}{x} : \frac{v^2}{y} : \frac{w^2}{z} = X_U^{\#} \end{array}$$

Proof. Direct computation. The symmetry between U, X is broken by using ϕ_U. □

Exercise 13.6.2. Explain why one obtains *sqrtdiv* $(U,\,P)$ and *sqrtdiv* (U, X), reverting the symmetry.

Proposition 13.6.3. *Start as described and use* $\psi_P = \boxed{P} . \boxed{\mathcal{C}^{-1}} . \boxed{P^{-1}}$. *This collineation is tailored so that* $\psi(\mathcal{T}_1) = ABC$, $\psi(\mathcal{T}_2) = \mathcal{A}_P$ *and* $\psi(\mathcal{T}_3)$ *is the anticevian of* $\psi(U)$ *wrt* $\mathcal{A}_P$. *Triangles* $\psi(\mathcal{T}_3)$ *and* $\psi(\mathcal{T}_2)$ *are perspective wrt* $\psi(U)$ *while* $\psi(\mathcal{T}_3)$ *and* $\psi(\mathcal{T}_1)$ *are perspective wrt* $\psi(X)$ *and :*

$$\psi \,.\, cevadiv\left(\psi^{-1} . P, \psi^{-1} . U\right) = sqrtdiv\,(P, U) = \frac{p^2}{u} : \frac{q^2}{v} : \frac{r^2}{w} = U_P^{\#}$$

Proposition 13.6.4. *Start as before, but use instead collineation* $(ABC, X_2) \mapsto (ceva\,(P), P)$ *i.e.* $\psi_P = \boxed{\mathcal{C}^{-1}} . \boxed{P^{-1}}$. *Then* $\psi(\mathcal{T}_1)$ *is* ABC *while* $\psi(\mathcal{T}_2)$ *is the anticomplementary triangle. Perspector between* $\psi(\mathcal{T}_1)$ *and* $\psi(\mathcal{T}_2)$ *is* $\psi(U) = X_2$, *perspector between* $\psi(\mathcal{T}_2)$ *and* $\psi(\mathcal{T}_3)$ *is* $\psi(P) = anticomplem\,(U/P)$ *while perspector between* $\psi(\mathcal{T}_1)$ *and* $\psi(\mathcal{T}_3)$ *is isotomic conjugate of the former. In other words :*

$$X = cevadiv\,(P, U) = \left(\psi^{-1} \circ isotom \circ \psi\right)(U)$$

13.7 Cevian conjugacies

Definition 13.7.1. The psi-Kimberling collineation of pole P is the collineation ψ_P such that $ABC \mapsto cevian\,(P)$ and $X_1 \to P$. Therefore :

$$\begin{aligned} \psi_P(U) &= P *_b complem\,(U \div_b X_1) \\ &= p\left(\frac{v}{b}+\frac{w}{c}\right) : q\left(\frac{u}{a}+\frac{w}{c}\right) : r\left(\frac{u}{a}+\frac{v}{b}\right) \\ \psi_P^{-1}(U) &= X_1 *_b anticomplem\,(U \div_b P) \\ &= a\left(-\frac{u}{p}+\frac{v}{q}+\frac{w}{r}\right) : b\left(\frac{u}{p}-\frac{v}{q}+\frac{w}{r}\right) : c\left(\frac{u}{p}+\frac{v}{q}-\frac{w}{r}\right) \end{aligned} \tag{13.2}$$

Remark 13.7.2. It is clear that ψ_P, ψ_P^{-1} are type-keeping when $X_1\,(a : b : c)$, $P(p : q : r)$ and $U(u : v : w)$ are transformed. Moreover, $\psi_P(A) = A_p = 0 : q : r$, $\psi_P(X_1) = P$ (from the very definition) while $\psi_P(-a : b : c) = A$ is obvious.

This ψ_P collineation has been used by Kimberling (2002a) to construct some new functions, following the patterns :

$$\phi \mapsto \psi_P \circ \phi \circ \psi_P^{-1} \quad or \quad \phi \mapsto \psi_P^{-1} \circ \phi \circ \psi_P$$

1. **cevadivision** of P by U can be re-obtained as $\psi_P \circ isogon \circ \psi_P^{-1}$. The result X is the perspector of *cevian* (P) and *anticevian* (U). More about this operation in Section 3.8. One has the formulas :

$$X = (-uqr + vrp + wqp)\,u \;:\; (uqr - vrp + wqp)\,v \;:\; (uqr + vrp - wqp)\,w$$
$$P = (vz + wy)^{-1} \;:\; (uz + wx)^{-1} \;:\; (yu + xv)^{-1}$$

   (a) cevadivision et ceva-multiplication are both type-keeping with respect to P and U. Using isogonal conjugacy result in the disappearing of a, b, c from the equations.

   (b) fixed points are $P = \psi_P(X_1)$ and the three vertices $A = \psi_P(-a : b : c)$, since $\pm a : \pm b : \pm c$ are the fixed points of *isogon*. A brute force resolution leads also to the cevians of P. A Taylor expansion around $1 : 0 : 0$ shows that vertices are really fixed points of cevadivision, while a Taylor expansion around $0 : p : q$ shows that undetermined $\psi_P(0 : q : r) = 0 : 0 : 0$ must be determined as $\psi_P(0 : q : r) = 0 : -q : r$

2. **alephdivision** of P by U : $\psi_P^{-1} \circ isogon \circ \psi_P$ (Hyacinthos #4111, Oct. 11, 2001). Formulas (cyclically) :

$$x \simeq a\left(p^2r^2v^2 + q^2p^2w^2 - q^2r^2u^2\right) + \frac{p^2r^2b^2 + q^2p^2c^2 - q^2r^2a^2}{bc}\,(vaw + ubw + cuv)$$
$$p^2 : q^2 : r^2 = \frac{1}{(bw+cv)(bz+cy)} \;:\; \frac{1}{(cu+aw)(cx+az)} \;:\; \frac{1}{(av+bu)(ay+bx)}$$

   Therefore, the alephmultiplication gives four result, one inside the triangle ABC and three outside

3. **bethdivision** of P by U : $\psi_P \circ sym3 \circ \psi_P^{-1}$ (Hyacinthos #4146, Oct. 26, 2001) where involution $sym3$ is the reflection in the circumcenter X_3 (15.9). This involution $sym3$ is related to the Darboux cubic. Barycentrics are :

$$\begin{aligned} x &\simeq a\,u - \frac{p\,(c+b)\,(a+c-b)}{q\,(-a+b+c)}\,v - \frac{p\,(c+b)\,(b+a-c)}{r\,(-a+b+c)}\,w \\ &= -a\,u + \frac{\widehat{p}}{\widehat{q}}\,(c+b)\,v + \frac{\widehat{p}}{\widehat{r}}\,(c+b) \qquad where\ \widehat{p} : \widehat{q} : \widehat{r} = P * X_7 \end{aligned}$$

   (a) This operation is type-keeping with respect to P, U.
   (b) The fixed points of $U \mapsto \beta\,(P,U)$ are obtained by ψ_P from the fixed points of $sym3$. They are $\psi_P\,(X_3)$ together with $\psi_P\,(\mathcal{L}_\infty)$, namely the line : $u/\widehat{p} + v/\widehat{q} + w/\widehat{r} = 0$.
   (c) Bethdivision of $X_{21} = a\,(b+c-a)\,/\,(b+c)$ by the circumcircle gives the circumcircle.
   (d) Bethdivision of P by U is P if and only if $U = P *_b X_{57}$.
   (e) Bethmultiplication is not simple (equation of third degree).

   Exercise 13.7.3. What are the situations where the discriminant vanishes ?

4. **gimeldivision** of P by U : $\psi_P^{-1} \circ sym3 \circ \psi_P$. Using barycentrics, one obtains :

$$\begin{aligned} F\,(U) &= 16\sigma^2\,U - \alpha\,\beta\,X_1 + 2\,\alpha\,(X_{48} *_b isot\,(P)) \quad where \\ 16\sigma^2 &= (b+c-a)\,(a+c-b)\,(b+a-c)\,(b+a+c) \\ \alpha &= \frac{(q+r)\,u}{a} + \frac{(r+p)\,v}{b} + \frac{(p+q)\,w}{c} \\ \beta &= \frac{\left(b^2+c^2-a^2\right)a^2}{p} + \frac{\left(c^2+a^2-b^2\right)b^2}{q} + \frac{\left(a^2+b^2-c^2\right)c^2}{r} \end{aligned}$$

   (a) The fixed points of $U \mapsto \gamma\,(P,U)$ are obtained by ψ_P^{-1} from the fixed points of $sym3$. They are $\psi_P^{-1}\,(X_3)$ together with $\psi_P\,(\mathcal{L}_\infty)$, namely the line : $u\,bc\,(q+r) + v\,ca\,(r+p) + w\,ab\,(p+q) = 0$.
   (b) Gimel multiplication leads to three points on the triangle sides, and three other points.

5. **mimosa** aka "much ado about nothing" X(1707)-X(1788). As with other names in ETC, the name Mimosa is that of a star. Define $mimosa\,(P)$ as $\psi_P^{-1}\,(X_3)$. Using barycentrics, one obtains :

$$mimosa\,(p:q:r) = u : v : w \qquad where$$
$$u = a\left(-\frac{\left(-a^2+b^2+c^2\right)a^2}{p} + \frac{b^2\left(a^2-b^2+c^2\right)}{q} + \frac{c^2\left(a^2+b^2-c^2\right)}{r}\right)$$

and also :

$$\begin{aligned} mimosa\,(P) &= cevadiv\,(X_{92} *_b P,\, X_1) \\ mimosa^{-1}\,(U) &= cevamul\,(U,\, X_1) *_b X_{63} \end{aligned}$$

Then, marvelously, the Mimosa transform M(X) arises in connection with the equation $gimeldiv(P,X) = X$. And there are too many such cases of gimel conjugates for all to be itemized in ETC... Here is a list of pairs (I,J) for which X(J) = M(X(I)).

1	46	20	1712	48	43	71	846	85	1729
2	19	21	4	54	47	72	191	86	1730
3	1	27	1713	55	1721	73	1046	88	1731
4	920	28	1714	56	1722	74	1725	89	1732
6	1707	29	1715	57	1723	75	1726	90	90
7	1708	31	1716	58	1724	77	57	95	92
8	1158	35	1717	59	109	78	40	96	91
9	1709	36	1718	60	580	80	1727	97	48
10	1710	37	1719	63	9	81	579	98	1733
19	1711	40	1720	69	63	84	1728	99	1577

6. **zosma**, yet another star. X(1824)-X(1907). The Zosma transform of a point X is the isogonal conjugate of the inverse mimosa transform of X.

7. **dalethdivision** of P by U : $\psi_P \circ hirst_1 \circ \psi_P^{-1}$ where $hirst_1\,(X) = hirstpoint\,(X_1,\, X)$ and thus $U \neq P$. Using barycentrics, one obtains :

$$x \simeq \left(\frac{w}{r} - \frac{v}{q}\right)^2 p - \left(\frac{u}{p} + \frac{v}{q} + \frac{w}{r}\right) u - 3\,\frac{u^2}{p}$$

   (a) This operation is type-keeping with respect to P, U.

   (b) The locus of fixed points of $hirst_1$ is the circumconic $cc\,(X_1)$. Therefore, the locus of fixed points of $daleth_P$ is the conic $cvc\,(P, P)$ tangent to the sidelines of ABC at the cevian points of P.

8. **hedivision** of P by U : $\psi_P^{-1} \circ \phi \circ \psi_P$ where $\phi\,(X) = hirstpoint\,(X_1,\, X)$ and thus $U \neq \psi_P^{-1}\,(P)$. Using barycentrics, one obtains :

$$x \simeq -p\left(\frac{v}{b} + \frac{w}{c}\right)^2 + \frac{qa}{b}\left(\frac{u}{a} + \frac{w}{c}\right)^2 + \frac{ra}{c}\left(\frac{u}{a} + \frac{v}{b}\right)^2 \\ + \frac{rqa^2}{cbp}\left(\frac{u}{a}+\frac{v}{b}\right)\left(\frac{u}{a}+\frac{w}{c}\right) - \frac{qcp}{br}\left(\frac{u}{a}+\frac{w}{c}\right)\left(\frac{v}{b}+\frac{w}{c}\right) - \frac{brp}{qc}\left(\frac{u}{a}+\frac{v}{b}\right)\left(\frac{v}{b}+\frac{w}{c}\right)$$

   (a) The locus of fixed points is a conic, but not a conic of cevians.

13.8 Miscelanous

13.8.1 Poles-of-lines and polar-of-points triangles

Definition 13.8.1. Polars-of-points triangle. Consider the general triangle $\mathcal{T}$ with vertices T_i, for $i = 1, 2, 3$. The may be degenerate associated polars-of-points triangle $\mathcal{T}_U = pntpoltri\,(\mathcal{T})$ is defined by vertices :

$$U_i = tripolar\,(T_{i+1}) \wedge tripolar\,(T_{i+2}) \tag{13.3}$$

where indexes are taken modulo 3.

Definition 13.8.2. Poles-of-lines triangle. Consider the general triangle $\mathcal{T}$ with vertices T_i, for $i = 1, 2, 3$. The may be degenerate associated poles-of-lines triangle $\mathcal{T}_P = linpoltri\,(\mathcal{T})$ is defined by vertices :

$$P_i = tripole\,(T_{i+1} \wedge T_{i+2}) \tag{13.4}$$

where indexes are taken modulo 3.

Lemma 13.8.3. *Let (t_{jk}) the barycentrics of the $\mathcal{T}$ vertices. Then :*

$$\det \mathcal{T}_U = (\det \mathcal{T}_{isot})^2 \;;\; \det \mathcal{T}_P = \frac{(\det \mathcal{T})^2\, \det \mathcal{T}_{isot}}{\prod_9 Adjoint\,(\mathcal{T})}$$

where $\mathcal{T}_{isot}$ is the triangle of the isotomics of the vertices and Π_9 is the condition expressing that two vertices of $\mathcal{T}$ are aligned with a vertex of ABC.

Proposition 13.8.4. *When $\mathcal{T}$ is flat (aligned points), then $\mathcal{T}_P$ is totally degenerate. When isotomic conjugates are collinear, $\mathcal{T}_P$ is flat, i.e. simply degenerate. Otherwise, $linpoltri\,(\mathcal{T})$ is a triangle. The line-polarity transform is type-keeping (and both tribes share the same formula).*

Proof. For example $U_2 \wedge U_3$ gives the barycentrics of line U_2U_3, while tripole is transpose and invert : being the product of two type-crossing transforms, $linpoltri$ is type-keeping. □

Remark 13.8.5. An example of flat line-polar triangle is given by triangles sharing the circumcircle of ABC.

Proposition 13.8.6. *When isotomic conjugates are collinear, $\mathcal{T}_U$ is totally degenerated. Otherwise pntpoltri $(\mathcal{T})$ is a triangle. Point-polarity is type-keeping (and both tribes share the same formula).*

Proof. For example, the polar of P_1 is the line $x/p_1 + y/q_1 + z/r_1 = 0$, and the polars of P_2 and P_3 are defined cyclically. Then U_1 is obtained as the common point of the last two lines. □

Proposition 13.8.7. *Line-polar and point-polar transforms are converse of each other... in the generic case. More precisely, pntpoltri $(linpoltri\,(\mathcal{T}))$ gives $\mathcal{T}$ times* $\det \mathcal{T}$*, going back to any non degenerate triangle. On the contrary, linpoltri $(pntpoltri\,(\mathcal{T}))$ gives $\mathcal{T}$ times* $1/\det \mathcal{T}_{isot}$*: the converse relation holds certainly when isotomic conjugates of points P_i aren't collinear* and *points P_i aren't collinear either* and *points P_i aren't on the sidelines.*

13.8.2 Unary cofactor triangle, eigencenter

Definition 13.8.8. The **unary cofactor triangle** of triangle U_i $(i = 1, 2, 3)$ is the triangle whose vertices are the isoconjugates of the vertices of the line-polar triangle of the points U_i. This operator is type-crossing over the U_i, but is nevertheless type-keeping over all involved points when using any fixed point F instead of $P = F^2$. Using barycentrics :

$$X_i = {}^t(U_{i+1} \wedge U_{i+2}) *_b P$$

where indexes are taken modulo 3.

Proposition 13.8.9. *When triangle $U_1U_2U_3$ is degenerate (collinear vertices), then triangle $X_1X_2X_3$ is totally degenerate (reduced to a point). Apart this situation, the unary cofactor transform is involutory.*

Definition 13.8.10. Eigencenter. Any triangle $U_1U_2U_3$ and its unary cofactor $X_1X_2X_3$ are perspective. Their perspector is called the eigencenter of these triangles (formula don't simplify, and has Maple-length 945).

When the original triangle is the cevian or the anticevian of a point U, formula shorten into :

$$\begin{aligned} eigencenter\,(\mathcal{C}_U) &= anticomplem\left((U *_b U)^*_P\right) *_b U^*_p = cevadiv\,(U, U^*) \\ eigencenter\,(\mathcal{A}_U) &= anticomplem\,(U *_b U \div_b P) *_b U \end{aligned}$$

These points are called, respectively, the eigentransform and the antieigentransform of point U (see Section 15.7).

Chapter 14

Isoconjugacies and Cremona group

14.1 Cross-ratio and fourth harmonic

Proposition 14.1.1. ***Cross-ratio***. *Suppose that we have chosen four different members of a linear projective familly. Let P, Q, R, S be the way they are written. Then non zero multipliers p, q, r, s and a constant λ can be found such that :*

$$\begin{array}{rcl} r\,R & = & p\,P + q\,Q \\ s\,S & = & p\,P + \lambda\,q\,Q \end{array}$$

Moreover quantity λ is well defined by the 4-uple, i.e. depends only on the four objects and their order. This quantity is called the cross-ratio of the 4-uple and written as

$$\lambda = \text{cross-ratio}\,(P, Q, R, S)$$

Proof. By definition of linearity, we have :

$$r_1 R = p_1 P + q_1 Q,\ s_2 S = p_2 P + q_2 Q$$

where none of the coefficients can be zero. Then $r = r_1 p_2$, $p = p_1 p_2$, $q = q_1 p_2$, $s = s_2 p_1$ and $\lambda = p_1 q_2 \div q_1 p_2$ is a solution. □

Theorem 14.1.2. *Cross-ratio is a projective object, i.e. is invariant by any collineation.*

Proof. Obvious from the definition. In fact, this result is the rationale for such a rather intricated definition. □

Proposition 14.1.3. *When the linear familly is parameterized from a pair of generators A, B by $P = k_P\,A + (1 - k_P)\,B$, etc then*

$$\text{cross-ratio}\,(P,\,Q,\,R,\,S) = \frac{(k_S - k_P)\,(k_R - k_Q)}{(k_S - k_Q)\,(k_R - k_P)} = \frac{(k_S - k_P)}{(k_S - k_Q)} \div \frac{(k_R - k_P)}{(k_R - k_Q)}$$

Proof. Assume $P \neq Q$. Then invert the parameterization, using :

$$A = \frac{(1 - k_Q)\,P - (1 - k_P)\,Q}{k_P - k_Q},\ B = \frac{k_P Q - k_Q P}{k_P - k_Q}$$ □

Proposition 14.1.4. *Let be given three members P, Q, R of a linear projective familly with, at least $P \neq Q$. Then it exists exactly one member S of the familly such that* $\text{cross-ratio}\,(P,\,Q,\,R,\,S) = -1$. *This object is called the fourth harmonic of the first three.*

Proof. Cross-ratio is an homographic function of parameter k_S, and therefore bijective between λ and k_S. □

Proposition 14.1.5. *All the lines λ through a given point P form a linear projective familly F. Consider a transversal line D (i.e. a line that doesn't go through P). Then cross ratio remains unchanged by application $F \hookrightarrow D$, $\lambda \mapsto \lambda \cap D$.*

Proof. Obvious since the wedge operator is a linear transform $\lambda \mapsto \lambda \wedge D$: parameterization is preserved. □

Proposition 14.1.6. *Consider four fixed points A, B, C, U with no alignments and define the moving cross-ratio of a point M in the plane as the cross-ratio of lines MA, MB, MC, MU. The level lines of this function are the conics passing through A, B, C, U. Using barycentrics with respect to ABC, we have the more precise statement: the level line of a given μ is the circumscribed conic whose perspector is $P = \mu u : -v : w\,(1-\mu)$, on the tripolar of $U = u : v : w$.*

Definition 14.1.7. Conic-cross-ratio. Consider a fixed, non degenerated conic $\mathcal{C}$, and four points A, B, C, U lying on this conic. Then the cross-ratio of lines MA, MB, MC, MU does not depends upon the choice of the auxiliary point M as long as this point M remains on the conic. When perspector $P = p : q : r$ of the circumconic is given, we have :

$$\text{cross-ratio}\,(A, B, C, U) = \mu \quad \text{when} \quad U \simeq \left(\begin{array}{c} (1-\mu)\,p \\ \left(\mu^2-\mu\right) q \\ \mu\, r \end{array} \right)$$

Remark 14.1.8. In the complex plane, the quantity defined in Proposition 14.1.1 is nothing but the usual cross-ratio, as computed from complex affixes. In order to be sure that, given four points on a circle, the conic cross-ratio and the usual cross-ratio are the same quantity, let us consider the stereographic projection. Start from $P\,(c, s)$. Define $M\,(C, S)$ by doubling the rotation and $T\,(0, t)$ by intersection of SM with the y-axis. Apply Thales to similar triangles SOT and SKM, and Pythagoras to OHP. This gives :

$$c^2 + s^2 = 1,\ (C,\, S) = \left(c^2 - s^2,\, 2cs\right),\ \frac{t}{1} = \frac{S}{1+C}$$

and leads to $t = s/c$, proving that $(SO, ST) = (ON, OP)$ and therefore that

$$\cos 2\vartheta = \frac{1-t^2}{1+t^2},\ \sin 2\vartheta = \frac{2t}{1+t^2}$$

Moreover, circular cross-ratio between M_j points is equal to linear cross-ratio between T_j points (this is the definition), while complex cross-ratio between M_j points is equal to complex cross-ratio between T_j points due to the homography :

$$z = (1 + i\,t)/(1 - i\,t)\,;\ t = i\,(1-z)/(1+z)$$

Proposition 14.1.9. *Group of the cross-ratio. The cross ratio remains unchanged under the action of the bi-transpositions like $[a, b, c, z] \mapsto [b, a, z, c]$. Therefore the action of $\mathfrak{S}_4$ generates only 6 values for the cross-ratio. We have :*

$$\begin{array}{cccccc} a & a & a & z & c & b \\ b & b & z & b & a & c \\ c & z & c & c & b & a \\ z & c & b & a & z & z \\ k & \dfrac{1}{k} & 1-k & \dfrac{k}{k-1} & \dfrac{k-1}{k} & \dfrac{1}{1-k} \end{array}$$

Special cases are $\{1, 0, \infty\}$ *(two points are equal),* $\{-1, 2, 1/2\}$ *(harmonicity) and* $\{\exp(+i\pi/3), \exp(-i\pi/3)\}$ *(equilateral triangle and one of its centers).*

Proof. Direct examination. □

14.2 Introducing the Cremona transforms

Definition 14.2.1. The upper spherical map is the projection $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}} \mapsto \mathbf{Z} : \mathbf{T}$, while the lower spherical map is the projection $\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}} \mapsto \overline{\mathcal{Z}} : \mathbf{T}$. Each of them sends $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ onto (yet another copy) of the Riemann sphere $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)$.

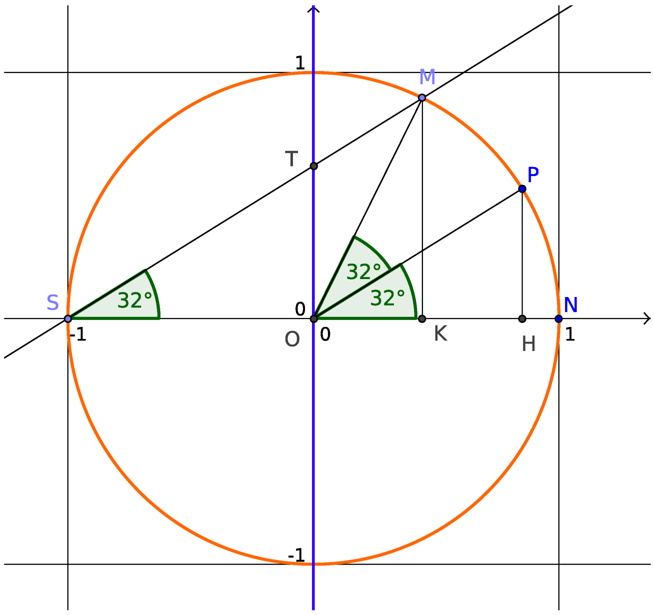

Figure 14.1: Stereographic projection

Theorem 14.2.2. *Consider a non-degenerated conic $\mathcal{C}$ in the projective plane and two fixed tangents Δ_1, Δ_2 to that conic. A moving tangent Δ cuts Δ_1 at M and Δ_2 at N. If we adopt two linear parameterizations, k on Δ_1 and K on Δ_2, the relation $M \mapsto N$ induces an homography between k and K. Moreover correspondence $\Delta_1 \hookrightarrow \Delta_2 : M \mapsto N$ can be extended into a transform Ψ that acts into $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ and looks like a pair of homographies ψ, $\overline{\psi}$, each of them acting onto one of the spherical maps.*

Proof. Consider the inscribed conic whose auxiliary point is $Q = u : v : w$. Consider lines AB, AC and parameterize by $M = k\,A + (1-k)\,B$, $N = K\,B + (1-K)\,C$. Assume that MN is tangent to $\mathcal{C}$ and obtain :

$$K = \frac{kw}{(u+v+w)\,k - v}$$

This proves the first part. Using the Lubin transmutation, we have :

$$M_z \doteq \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix} \simeq \boxed{aller} \cdot \begin{pmatrix} k \\ 1-k \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha\,k + \beta\,(1-k) \\ 1 \\ \dfrac{k}{\alpha} + \dfrac{1-k}{\beta} \end{pmatrix}$$

$$N_z \doteq \begin{pmatrix} \mathbf{Z}' \\ \mathbf{T}' \\ \overline{\mathcal{Z}}' \end{pmatrix} \simeq \boxed{aller} \cdot \begin{pmatrix} K \\ 0 \\ 1-K \end{pmatrix} \simeq \begin{pmatrix} \alpha\,kw + \gamma\,(ku+vk-v) \\ k\,(u+v+w) - v \\ \dfrac{kw}{\alpha} + \dfrac{ku+vk-v}{\gamma} \end{pmatrix}$$

Indentifying with respect to parameter k, we are conducted to define

$$\boxed{\psi} = \begin{pmatrix} (\gamma\,(u+v) + \alpha\,w) & -(\beta\gamma\,u + \gamma\alpha\,v + \alpha\beta\,w) \\ u+v+w & -(\alpha\,v + \beta\,(u+w)) \end{pmatrix}$$

in order to have :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Z}' \\ \mathbf{T}' \\ \overline{\mathcal{Z}}' \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} & 0 \\ \psi_{21} & \psi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{Z}' \\ \mathbf{T}' \\ \overline{\mathcal{Z}}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{\psi_{22}} & \overline{\psi_{21}} \\ 0 & \overline{\psi_{12}} & \overline{\psi_{11}} \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{pmatrix}$$

□

Theorem 14.2.3. *(Continued). Finally, the four focuses of $\mathcal{C}$ are the fixed points of the transform Ψ that acts into $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ while the projections of the focuses onto the upper spherical map are the two fixed points of $\psi \in \mathbb{PGL}_{\mathbb{C}}\left(\mathbb{C}^2\right)$.*

Proof. Fixed points of ψ are the roots of :

$$(u+v+w)\,\mathbf{Z}^2 - (\alpha\,(v+w) + \beta\,(w+u) + \gamma\,(u+v))\,\mathbf{ZT} + (\beta\gamma\,u + \gamma\alpha\,v + \alpha\beta\,w)\,\mathbf{T}^2 = 0$$

This equation can be rewritten into :

$$\frac{u}{\mathbf{Z}-\alpha\mathbf{T}} + \frac{v}{\mathbf{Z}-\beta\mathbf{T}} + \frac{w}{\mathbf{Z}-\gamma\mathbf{T}} = 0$$

and this equation characterizes the focuses of the conic in the upper map. □

Lemma 14.2.4. *Let ψ be an element of $\mathbb{PGL}_{\mathbb{C}}\left(\mathbb{C}^2\right)$, i.e. an homography $z \mapsto (az+b)\,/\,(cz+d)$ of the Riemann sphere. If we assume that the fixed points f_1, f_2 of ψ are different, then ψ is characterized by a number $\mu \in \mathbb{C}$ that is neither 0 nor ∞. We call it the* ***multiplier*** *of ψ and we have :*

$$\mu = \text{cross-ratio}\,(f_1, f_2, z, \psi\,(z)) = \frac{c\,f_2 + d}{c\,f_1 + d} = \psi'\,(f_1) = \frac{1}{\psi'\,(f_2)}$$

Proof. To see that $\text{cross-ratio}\,(f_1, f_2, z_1, \psi\,(z_1)) = \text{cross-ratio}\,(f_1, f_2, z_2, \psi\,(z_2))$, use the fact that $\text{cross-ratio}\,(f_1, f_2, z_1, z_2)$is invariant. □

Lemma 14.2.5. *A more symmetrical quantity is :*

$$\sigma \doteq \mu + \frac{1}{\mu} - 2 = \frac{(\mu-1)^2}{\mu} = \frac{(a-d)^2 + 4\,b\,c}{a\,d - b\,c}$$

Proof. Direct computation. □

Remark 14.2.6. An homography $z \mapsto (az+b)\,/\,(cz+d)$ of the Riemann sphere is involutory when $a+d=0$.

Proposition 14.2.7. *The focal transform :*

$$\mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}} \mapsto \frac{(f_1 - \mu\,f_2)\,\mathbf{Z} + (\mu-1)\,f_1 f_2\,\mathbf{T}}{(1-\mu)\,\mathbf{Z} + (\mu\,f_1 - f_2)\,\mathbf{T}} : 1 : \frac{(g_1 - \nu\,g_2)\,\overline{\mathcal{Z}} + (\nu-1)\,g_1 g_2\,\mathbf{T}}{(1-\nu)\,\overline{\mathcal{Z}} + (\nu\,g_1 - g_2)\,\mathbf{T}}$$

is transmuted into the following standardized transform :

$$\Psi_0 \doteq \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}} \mapsto \frac{(1+\mu)\,\mathbf{Z} + (1-\mu)\,\mathbf{T}}{(1-\mu)\,\mathbf{Z} + (1+\mu)\,\mathbf{T}} : 1 : \frac{(1+\nu)\,\overline{\mathcal{Z}} + (1-\nu)\,\mathbf{T}}{(1-\nu)\,\overline{\mathcal{Z}} + (1+\nu)\,\mathbf{T}}$$

by similitude :

$$\left(\begin{array}{ccc} f_1 - f_2 & f_1 + f_2 & 0 \\ 0 & 2 & 0 \\ 0 & g_1 + g_2 & g_1 - g_2 \end{array}\right)$$

while Ψ_0 can be factored into :

$$\Psi_0 = trans \circ \sigma \circ multi \circ trans$$

where the Cremona transform σ, the multiplication and the translation (the same translation is applied once and again, not once and the reverse afterwards) are defined by :

$$\sigma \left(\begin{array}{c} \mathbf{Z} \\ \mathbf{T} \\ \overline{\mathcal{Z}} \end{array}\right) \simeq \left(\begin{array}{c} 1/\mathbf{Z} \\ 1/\mathbf{T} \\ 1/\overline{\mathcal{Z}} \end{array}\right), \quad \boxed{multi} = \left(\begin{array}{ccc} \sigma_\mu & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \sigma_\nu \end{array}\right), \quad \boxed{trans} = \left(\begin{array}{ccc} 1 & \dfrac{\mu+1}{\mu-1} & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{\nu+1}{\nu-1} & 1 \end{array}\right)$$

Proof. Direct computation. □

Definition 14.2.8. The Cremona group is defined as the set of the birational transforms of $\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$. Therefore a transformation $\Psi \in$ Cremona can be written as :

$$\Psi\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)=\psi_1\left(\mathbf{Z},\mathbf{T},\overline{\mathcal{Z}}\right):\psi_2\left(\mathbf{Z},\mathbf{T},\overline{\mathcal{Z}}\right):\psi_3\left(\mathbf{Z},\mathbf{T},\overline{\mathcal{Z}}\right)$$

where the ψ_j are three homogeneous polynomials of the same degree. And the existence of another transform $\Phi \in$ Cremona is assumed so that, at least formally, $(\Phi\circ\Psi)\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)\simeq\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)$.

Exercise 14.2.9. What can be said about the degrees when two transforms are inverse of each other ? See Diller (2011)

Definition 14.2.10. Given a Cremona transform, we define the *indeterminacy points* and the *exceptional curves* by :

$$\begin{aligned}\operatorname{Ind}(\Psi) &= \{M \mid \psi_1(M)=\psi_2(M)=\psi_3(M)=0\}\\ \operatorname{Exc}(\Psi) &= \left\{M \,\middle|\, \det\frac{\partial(\psi_1,\,\psi_2,\,\psi_3)}{\partial\left(\mathbf{Z},\,\mathbf{T},\,\overline{\mathcal{Z}}\right)}=0\right\}\end{aligned}$$

Proposition 14.2.11. *A quadratic transformation $f\in$ Cremona always acts by blowing up three (indeterminacy) points $\operatorname{Ind}(f)=\left\{p_1^+,\,p_2^+,\,p_3^+\right\}$ and blowing down the (exceptional) lines joining them. Typically, the points and the lines are distinct, but they can occur with multiplicity. Then f^{-1} is also a quadratic transformation and $\operatorname{Ind}\left(f^{-1}\right)=\left\{p_1^-,\,p_2^-,\,p_3^-\right\}$ consists of the images of the three exceptional lines.*

Proof. The relations between indeterminacy points and exceptional curves does not hold for higher degree transforms. □

Theorem 14.2.12. *The Cremona group is generated by collineations and the "inverse everything" transform i.e. $\sigma:\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)\mapsto\left(\mathbf{T}\overline{\mathcal{Z}}:\overline{\mathcal{Z}}\mathbf{Z}:\mathbf{Z}\mathbf{T}\right)$.*

Proof. A detailled proof can be found in Alberich-Carramiñana (2002), and an historical sketch is given in Déserti (2009). The idea is to separate infinitely neighbor points in $\operatorname{Ind}(\psi)$ if required and then proceed to a descending recursion over the cardinal of $\operatorname{Ind}(\psi)$. □

14.3 Working out some examples

Example 14.3.1. Transform ρ is $\rho\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)=\left(\mathbf{Z}\overline{\mathcal{Z}}:\mathbf{T}\overline{\mathcal{Z}}:\mathbf{T}^2\right)$. This is an involution. The set $\operatorname{Ind}(\rho)$ contains $1:0:0$ (twice) and $0:0:1$. The exceptional locus is the reunion of the contraction lines $\mathbf{T}=0$ and $\overline{\mathcal{Z}}=0$. The decomposition of this transform can be conducted as described in Table 14.1.

Example 14.3.2. Transform μ is $\mu\left(\mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}\right)=\left(\mathbf{Z}\overline{\mathcal{Z}}:\mathbf{Z}^2-\mathbf{T}\overline{\mathcal{Z}}:\overline{\mathcal{Z}}^2\right)$. This is an involution. The points of indeterminacy form a sequence of infinitely close neighbor $\mu_3^+\ll\mu_2^+\ll\mu_1^+=0:1:0$. Similarily, the exceptional locus is line $\overline{\mathcal{Z}}=0$, counted three times. Due to this specificity, its Cremona factorisation is longer. A description of this process is given in Déserti (2008–2009), leading to a nine steps process $(\phi_5\sigma\phi_4\sigma\phi_3\sigma\phi_2\sigma\phi_1)$.

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14.4 Isoconjugacy and sqrtdiv operator

Definition 14.4.1. An heuristic definition of the **sqrtdiv** operator was already given in Definition 1.5.10. Start from triangle ABC and consider a fixed point F whose ABC barycentrics are $f:g:h$. Then $U_F^{\#}$ is defined by :

$$sqrtdiv_F(U)\doteq U_F^{\#}\doteq\frac{f^2}{u}:\frac{g^2}{v}:\frac{h^2}{w} \tag{14.1}$$

Remark 14.4.2. This operator is globally type-keeping and therefore *sqrtdiv* is a pointwise transform. Its fixed points are the four $\pm f:\pm g:\pm h$, i.e. F and its associates under the Lemoine transforms.

	result	Ind (ψ)	transform
1	$\begin{pmatrix} xz \\ yz \\ y^2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
2	$\begin{pmatrix} (x+y)\,z \\ yz \\ (y-z)\,y \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	σ
3	$\begin{pmatrix} (y-z)\,y \\ (x+y)\,(y-z) \\ (x+y)\,z \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
4	$\begin{pmatrix} (y-z)\,y \\ (x+y)\,(y-z) \\ (x+y)\,y \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix}$	σ
5	$\begin{pmatrix} x+y \\ y \\ y-z \end{pmatrix}$	none	$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$
6	$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$	none	

Table 14.1: Reduction of a Cremona transform

Theorem 14.4.3. Formal definition of the sqrt operator. *Start from four independent points F_1, F_2, F_3, F_4 (three of them are never on the same line). For any point U in the plane, draw both conics :*

$$\mathcal{C}_{12} \doteq \mathcal{C}\left(U,\ F_1,\ F_2,\ F_1F_3 \cap F_2F_4,\ F_1F_4 \cap F_2F_3\right)$$
$$\mathcal{C}_{34} \doteq \mathcal{C}\left(U,\ F_3,\ F_4,\ F_1F_3 \cap F_2F_4,\ F_1F_4 \cap F_2F_3\right)$$

Their fourth intersection is independent of the order chosen for the set $\{F_j\}$ and is the sqrtdiv$_F$ image of U that was former defined wrt triangle $A = F_1F_2 \cap F_3F_4$,$B = F_1F_3 \cap F_2F_4$, $C = F_1F_4 \cap F_2F_3$.

Proof. Choose an order over set $\{F_j\}$, use the above defined triangle ABC as the reference triangle and let $f : g : h$ be the barycentrics of F_1 in this context. Then F_2, F_3, F_4 is the anticevian triangle of F_1 wrt ABC, enforcing $F_2 = -f : g : h$, etc. Compute the conics using the usual 6×6 determinant and obtain :

$$\boxed{\mathcal{C}_{12}} \simeq \left[\begin{array}{ccc} 2\,(gw-hv)\,ghu & \left(-f^2vw+ghu^2\right)h & \left(+f^2vw-ghu^2\right)g \\ \left(-f^2vw+ghu^2\right)h & 0 & (hv-gw)\,f^2u \\ \left(+f^2vw-ghu^2\right)g & (hv-gw)\,f^2u & 0 \end{array}\right]$$

$$\boxed{\mathcal{C}_{34}} \simeq \left[\begin{array}{ccc} 2\,(gw+hv)\,ghu & \left(-f^2vw-ghu^2\right)h & \left(-f^2vw-ghu^2\right)g \\ \left(-f^2vw-ghu^2\right)h & 0 & (hv+gw)\,f^2u \\ \left(-f^2vw-ghu^2\right)g & (hv+gw)\,f^2u & 0 \end{array}\right]$$

The result is straightforward, and its symmetry implies the independence from the way set $\{F_j\}$ was ordered. □

Corollary 14.4.4. *Any sqrtdiv operator is an involution. When U is on line F_jF_k, so is $U_F^{\#}$ and we have :*

$$birap\left(F_j,\, F_k,\, U,\, U_F^{\#}\right) = -1$$

Proof. We have : $\det\left(F_1,\, F_2, U_F^{\#}\right) = (-gh/vw) \times \det\left(F_1,\, F_2,\, U\right)$. □

Construction 14.4.5. *The fourth harmonic (harmonic conjugate of point U wrt points F_1F_2) can be constructed by chosing two arbitrary points F_3F_4 and then using the sqrtdiv operator having the four F_j as fixed points. If we chose F_3, F_4 in order to obtain the middle of F_1F_2 and the two umbilics as triangle ABC, we obtain the usual construction using one circle from the pencil admitting F_1F_2 as base points and one from the pencil admitting F_1F_2 as limit points.*

Definition 14.4.6. Isoconjugacy. Forget that $f^2 : g^2 : h^2$ are the square of the barycentrics of four points, and consider them as the barycentrics of a new point, called the pole $P = p : q : r$ of the transform. Then the isoconjugate of $U = u : v : w$ wrt pole P is obtained by :

$$U_P^* = (u : v : w)_P^* \doteq \frac{p}{u} : \frac{q}{v} : \frac{r}{w} \tag{14.2}$$

Remark 14.4.7. This transform was introduced in order to unify isotomic conjugacy (Section 3.3) and isogonal conjugacy into a common frame. When using barycentrics, isotomic conjugacy is obtained with $P = X_2$ and isogonal conjugacy with $P = X_6$. When using trilinears, you have to use (respectively) $P = X_{75}$ and $P = X_1$. When X_2 is special, its isogonal conjugate X_6 is special too. When $X_1$is special, its isotomic conjugate $X_{75}$is special too.

Corollary 14.4.8. *Isoconjugacy $U \mapsto U^*$ is type-crossing (respective to U alone) and involutory. This mapping has four fixed points (real or not), namely the points $F_i \doteq \pm\sqrt{p} : \pm\sqrt{q} : \pm\sqrt{r}$.*

Construction 14.4.9. *Let be given two points $U \neq V$, not on the sidelines and such that UV doesn't go through a vertex. Then the isoconjugacy ψ that exchanges UV can be constructed as follows. Once for ever, point R is chosen on line UV and its cevian triangle $A_RB_RC_R$ is obtained, and triangle $\mathcal{T}_1$ is constructed as :*

$$A_1 = VA \cap UA_R,\; B_1 = VB \cap UB_R,\; C_1 = VC \cap UC_R$$

Then consider a point X not on the sidelines, define triangle $\mathcal{T}_2$ by $A_2 = A_1X \cap BC$ (etc) and triangle $\mathcal{T}_3$ as cross $(\mathcal{T}_1, \mathcal{T}_2)$ i.e. $A_3 = B_1C_2 \cap C_1B_2$ (etc). It happens that triangle ABC and $A_3B_3C_3$ are perspective, and this perspector is the required point $\psi(X)$.

Proof. Put $V = p : q : r$ and $U = u : v : w$. Express R as $R = V - \rho U$. Existence of A_1 requires $wq - vr \neq 0$, i.e. A, U, V not colinear. The result is :

$$\mathcal{T}_1 = \left(\begin{array}{ccc} \rho u & p & p \\ q & \rho v & q \\ r & r & \rho w \end{array}\right), \qquad \mathcal{T}_2 = \left(\begin{array}{ccc} 0 & \rho vx - py & \rho wx - pz \\ \rho uy - qx & 0 & \rho wy - qz \\ \rho uz - rx & \rho vz - ry & 0 \end{array}\right)$$

$$\mathcal{T}_3 = \left(\begin{array}{ccc} p\,(yw + zv) - \rho vwx & puz & puy \\ qvz & q\,(zu + xw) - \rho wuy & qxv \\ rwy & rwx & r\,(xv + yu) - \rho uvz \end{array}\right)$$

and the perspector is : $pu/x : qv/y : rw/z$ as required. In (Dean and van Lamoen, 2001), ψ was called **reciprocal conjugacy**. □

Construction 14.4.10. *Another construction, with same hypotheses. Call shadows of X the reintersections X_a, X_b, X_c of lines AX, BX, CX and the conic γ that goes through A, B, C, U, V. Define the traces of UV as $T_a \doteq UV \cap BC$,* etc. *Then traces and shadows are collinear, i.e. Y_a is the second intersection of T_aX_a with γ,* etc.

Proof. Direct computation. Here again $wq - vr \neq 0$, etc is required. □

Remark 14.4.11. Most of the time, barycentric multiplication appears as the result of two successive isoconjugacies, according to :

$$\left(X_U^*\right)_P^* = X *_b U_P^*$$

For example, $isot\left(isog\left(X\right)\right) = \left(X_6^*\right)_2^* = X *_b isot\left(X_6\right)$ while $isog\left(isot\left(X\right)\right) = \left(X_2^*\right)_6^* = X *_b isog\left(X_2\right)$

Remark 14.4.12. Maps *sqrtmul* and *sqrtdiv* are related with cevian nests (Table 3.2, case III). This leads to another construction of $\psi\left(X\right)$ when $P = U = \psi\left(P\right)$ is given.

14.4.1 Morley point of view

Proposition 14.4.13. Isogonal conjugacy. *The Morley affix of the isogonal conjugate of point $P = \mathbf{Z} : \mathbf{T} : \overline{\mathcal{Z}}$ is given by :*

$$isog\begin{pmatrix}\mathbf{Z}\\ \mathbf{T}\\ \overline{\mathcal{Z}}\end{pmatrix} \simeq \begin{pmatrix}\sigma_3\overline{\mathcal{Z}}^2 - \mathbf{ZT} - \sigma_2\,\overline{\mathcal{Z}}\mathbf{T} + \sigma_1\,\mathbf{T}^2\\ \mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}}\\ \dfrac{1}{\sigma_3}\,\mathbf{Z}^2 - \dfrac{\sigma_1}{\sigma_3}\,\mathbf{ZT} - \overline{\mathcal{Z}}\mathbf{T} + \dfrac{\sigma_2}{\sigma_3}\,\mathbf{T}^2\end{pmatrix} \tag{14.3}$$

Proof. This formula can be stated using the representation of first degree. Start from point P and obtain Δ_A, the A-isogonal of line AP, by solving :

$$\begin{aligned}\Delta_A\,.\,A &= 0\\ \tan\left(AB,\,\Delta_A\right) + \tan\left(AC,\,AP\right) &= 0\end{aligned}$$

Compute $\Delta_A \wedge \Delta_B$ and obtain a symmetric expression, proving that Δ_C goes also through this point. One can check that $isog\left(\Omega^+\right) = \Omega^-$ and vice versa. □

Proposition 14.4.14. *The inexcenters are enumerated by the polynomial :*

$$\zeta^4 - 2\,\sigma_2\,\zeta^2 + 8\,\sigma_3\,\zeta + \left(\sigma_2^2 - 4\,\sigma_1\,\sigma_3\right) \tag{14.4}$$

Proof. The fixed points of the isogonal transform are obtained by solving $\mathbf{T}\,isog\left(M\right) - \left(\mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}}\right)M = 0$. This gives two polynomials of degree 3, that are not self-conjugate but conjugate of each other. The corresponding algebraic curves are not visible. Intersecting these curves, we obtain nine points. Among them, are both umbilics. The umbilical pair is fixed, but a given umbilic is not fixed. They are nevertheless appearing since they anulate both $\mathbf{T}$ and $\mathbf{T}^2 - \mathbf{Z}\overline{\mathcal{Z}}$. When computing the $\overline{\mathcal{Z}}$ resultant of these polynomials, we obtain an eight degree polynomial that factors into :

$$\mathbf{T} \times \prod_3 \left(\mathbf{Z} - \alpha\mathbf{T}\right) \times poly_4\left(\mathbf{Z},\,\mathbf{T}\right)$$

The umbilical pair is represented by $\mathbf{T}$, vertices are appearing and it remains the required polynomial of degree 4: this gives the $2 + 3 + 7 = 9$ intersections of two degree 3 curves. To be sure of what happens, we can compute the $\mathbf{T}$ resultant of both polynomials and obtain :

$$\mathbf{Z}\overline{\mathcal{Z}} \times \prod_3 \left(\mathbf{Z} - \alpha^2\overline{\mathcal{Z}}\right) \times poly_4\left(\mathbf{Z},\,\overline{\mathcal{Z}}\right)$$

where each point is represented by a specific factor. □

Remark 14.4.15. When substituting $\alpha = \alpha^{2}$, etc (i.e. using the Lubin representation of second degree), polynomial (14.4) splits, with roots $\pm\beta\gamma \pm \gamma\alpha \pm \alpha\beta$ as required.

14.4.2 Isoconjkim

In the old ancient times, Kimberling (1998) introduced another definition of the isoconjugacy, in an attempt to unify isotomic and isogonal conjugacies into a broader concept. Thereafter, this definition was changed into the one given above. To avoid confusion with (14.2), we will use the term **isoconjkim** to describe the older concept.

Definition 14.4.16. isoconjkim. For points outside the sidelines of ABC, the Kimberley P-**isoconjkim** of U is the point X such the product of the trilinears of P, U, X gives $1:1:1$. Restated into barycentrics, this gives :

$$P *_b U *_b X = X_1 *_b X_1 *_b X_1 \tag{14.5}$$

In this transformation, P and U play the same role so that *isoconjkim* acts like a multiplication. On the other hand U and X play also the same role, so that $isoconjkim_P$ is involutory. Description $X = isoconjkim_P(U)$ reflects the fact that second point of view is the more useful.

Example 14.4.17. Here is a list of various isoconjkim transformations. The *isogonal conjugation* is X_1-*isoconjkim* while the *isotomic conjugation* is X_{31}- *isoconjkim.*

P	barycentrics	trilinears	pole	bar(pole)	fixed
$X(1)$	$a^2\,\frac{1}{u}$	$1\,\frac{1}{u}$	$X(6)$	a^2	$X(1)$
$X(2)$	$a^3\,\frac{1}{u}$	$a\,\frac{1}{u}$	$X(31)$	a^3	$X(365)$
$X(3)$	$\left(a^2/\cos A\right)\frac{1}{u}$	$(1/\cos A)\,\frac{1}{u}$	$X(19)$	a/S_a	$X(???)$
$X(4)$	$a^2\cos A\,\frac{1}{u}$	$\cos A\,\frac{1}{u}$	$X(48)$	a^3S_a	$X(???)$
$X(6)$	$a\,\frac{1}{u}$	$(1/a)\,\frac{1}{u}$	$X(1)$	a	$X(366)$
$X(19)$	$a\,\cos A\,\frac{1}{u}$	$(\cos(A)/a)\,\frac{1}{u}$	$X(3)$	a^2S_a	$X(???)$
$X(31)$	$1\,\frac{1}{u}$	$\left(1/a^2\right)\frac{1}{u}$	$X(2)$	1	$X(2)$
$X(48)$	$(a/\cos A)\,\frac{1}{u}$	$1/(a\,\cos A)\,\frac{1}{u}$	$X(4)$	$1/S_a$	$X(???)$

14.5 Antigonal conjugacy

Proposition 14.5.1. Antigonal conjugacy. *Let A', B', C' be the reflections of a point M in the sidelines BC, CA, AB of the reference triangle. Then circles $ABC', AB'C, A'BC$ concur into a common point N (see Figure 14.2). This point N is called the antigonal conjugate of M. When point M is given either by $M_b \simeq p:q:r$ (barycentrics) or by $M_z \simeq \mathbf{Z}:\mathbf{T}:\overline{\mathcal{Z}}$ (Lubin affixes), point N is given by :*

$$N_b \simeq \left(\begin{array}{c} \dfrac{p}{(a^2-b^2-c^2)\,p^2+(a^2-b^2)\,pq+(a^2-c^2)\,pr+a^2qr} \\ \dfrac{q}{(b^2-c^2-a^2)\,q^2+(b^2-c^2)\,qr+(b^2-a^2)\,qp+b^2rp} \\ \dfrac{r}{(c^2-a^2-b^2)\,r^2+(c^2-a^2)\,rp+(c^2-b^2)\,rq+c^2pq} \end{array}\right)$$

$$N_z \simeq \left(\begin{array}{c} \dfrac{-\sigma_3\mathbf{Z}\overline{\mathcal{Z}}^2+\sigma_2\mathbf{Z}\overline{\mathcal{Z}}\mathbf{T}+\sigma_3\sigma_1\overline{\mathcal{Z}}^2\mathbf{T}-\sigma_1\mathbf{Z}\mathbf{T}^2+(\sigma_3-\sigma_1\sigma_2)\,\overline{\mathcal{Z}}\mathbf{T}^2+\left(\sigma_1^2-\sigma_2\right)\mathbf{T}^3}{\sigma_3\overline{\mathcal{Z}}^2-\mathbf{Z}\mathbf{T}-\sigma_2\overline{\mathcal{Z}}\mathbf{T}+\sigma_1\mathbf{T}^2} \\ \mathbf{T} \\ \dfrac{-\sigma_3\mathbf{Z}^2\overline{\mathcal{Z}}+\sigma_2\mathbf{T}\mathbf{Z}^2+\mathbf{T}\sigma_1\sigma_3\mathbf{Z}\overline{\mathcal{Z}}-(\sigma_1\sigma_2-\sigma_3)\,\mathbf{T}^2\mathbf{Z}-\sigma_2\sigma_3\overline{\mathcal{Z}}\mathbf{T}^2+\left(\sigma_2^2-\sigma_1\sigma_3\right)\mathbf{T}^3}{\sigma_3\left(\mathbf{Z}^2-\sigma_1\mathbf{Z}\mathbf{T}-\sigma_3\overline{\mathcal{Z}}\mathbf{T}+\sigma_2\mathbf{T}^2\right)} \end{array}\right)$$

Proof. Direct computation for N_b, then usual transform for N_z. □

Proposition 14.5.2. *The conic that goes through $A, B, C, M, N = antigon(M)$ is a rectangular hyperbola (and therefore goes through $H = X(4)$). Moreover M, N are antipodes on this conic.*

Proof. Direct computation. This suggests that antigonal transform is involutory. □

Proposition 14.5.3. *The antigonal transform can be factored into :*

$$antigon = isogon \circ invincircum \circ isogon$$

This is an involutory Cremona transform. The indeterminacy set contains six points: A, B, C, H and both umbilics. The exceptional locus is the reunion of six conics. Each of them goes through five points of $\mathrm{Ind}\,(antigon)$*: circumcircle is blown-down into H, circle BCH (centered at $B+C-O$) is blown-down into A, idem for the other two vertices. Finaly, curve :*

$$\gamma_y = \mathbf{Z}^2-\sigma_1\mathbf{Z}\mathbf{T}-\sigma_3\overline{\mathcal{Z}}\mathbf{T}+\sigma_2\mathbf{T}^2$$

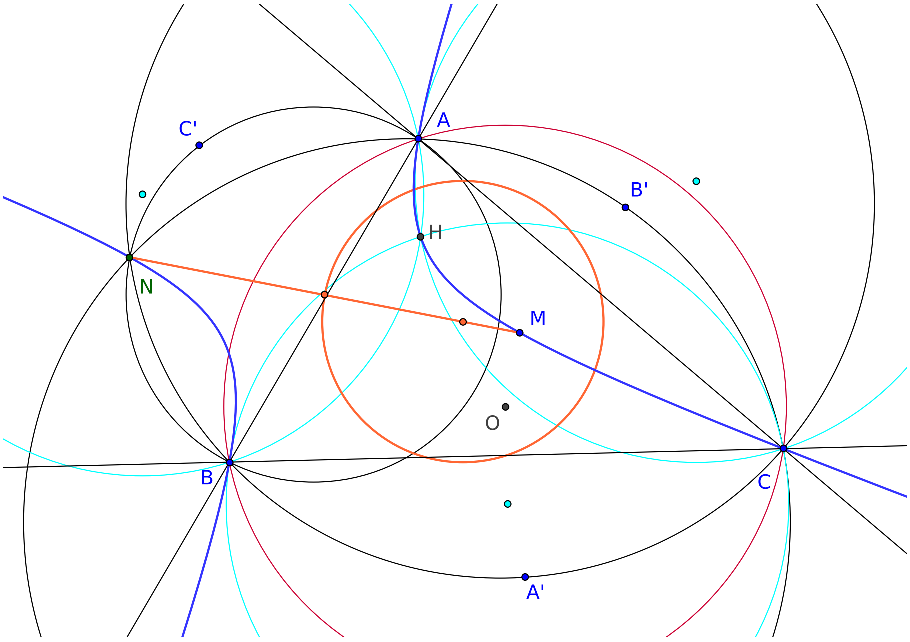


Figure 14.2: Antigonal conjugacy

that goes through A, B, C, H, Ω_y is blown-down into Ω_y (the same umbilic), idem for the other umbilic.

Proof. Direct computation. □

Proposition 14.5.4. *Define the ii transform as $ii = invincircum \circ isogon$ then point $ii\,(A, B, C, P)$ is independent of the ordering of points A, B, C, P (Hyacinthos 20929).*

Proof. Transform triangle ABC into ABP, and thus C into P'. Then use the usual change of triangle formula. Since ii only involves even powers of the sidelength, everything goes fine... and the result follows. □

Proposition 14.5.5. *Seen as a Cremona map, the ii transform has the same indeterminacy locus as the antigonal transform. The exceptional locus contains three lines and three conics. A sideline like BC blows-down into the opposite vertex A. Curve γ_x (the same as before) now blows down to Ω_y, whileγ_y blows-down to Ω_x. Additionally, the circumcircle blows-down to its center X(3), while $X\,(3) \underset{ii}{\mapsto} X(186)$ is regular and X(265) is the only regular point that maps onto H =X(4).*

Proof. This lack of symmetry (a curve that blows down to a regular point, a point of indeterminacy that does not blows-up under the reverse transform) is related to the fact that ii is not involutory. □

Chapter 15

About cubics

15.1 Cubic

For a catalog with sketches, visit Gibert (2004-2010), especially Ehrmann and Gibert (2009). Some notations :

I	G	O	H	N	K	L			
X(1)	X(2)	X(3)	X(4)	X(5)	X(6)	X(20)			

Definition 15.1.1. A cubic is a curve defined by an homogeneous polynomial of degree 3. Using barycentrics, or trilinears or Morley affixes is irrelevant, the degree is the same. Notation : $\mathcal{K}$.

Proposition 15.1.2. *A cubic is defined by nine general points.*

Proof. There are ten coefficients, defined up to a global proportionality factor. We found them computing

$$\det_{j=10}\left[x^3, x^2y, xy^2, y^3, y^2z, yz^2, z^3, z^2x, zx^2, xyz\right] \tag{15.1}$$

applied to the nine given points and the generic point. □

Theorem 15.1.3. Pivotal isocubics. *Let F, U be two points not on the sidelines and define $p\mathcal{K}\left(\#F,U\right)$ as the cubic that goes through ABC, $A_UB_UC_U$ (the cevians of U) and $F_AF_BF_C$ (the anticevians of F). Then U and $F_0 \doteq F$ are also on the cubic. Its equation is :*

$$p\mathcal{K}\left(P,U\right) \doteq p\mathcal{K}\left(\#F,U\right) \doteq \left(h^2y^2 - g^2z^2\right)ux + \left(f^2z^2 - h^2x^2\right)vy + \left(g^2x^2 - f^2y^2\right)wz \tag{15.2}$$

Let $X_F^{\#}$ be the isoconjugate of X wrt the four fixed points F_j. Then $p\mathcal{K}\left(\#F,U\right)$ is the locus of the X such that U, X, $X_F^{\#}$ are collinear. For this reason, point U is called the **pivot** *of the cubic. When X is on the cubic, then point cevadiv $\left(U,X\right)$ is the third intersection of the line $X\,U_F^{\#}$ with the cubic.*

Proof. Compute the determinant of the 10 rows (15.1) relative to the nine given points and the variable point $X = x : y : z$. Check that F_0 and U satisfies also the equation. Remark that this equation is proportionnal to $\det\left(U,\, X,\, X_F^{\#}\right)$. □

Proposition 15.1.4. *Define an isocubic $\mathcal{K}$ with pole P as a cubic that is invariant wrt the P isoconjugacy. Then $\mathcal{K}$ is either a pivotal cubic as obtained just above or belongs to the "non pivotal isocubic" family (Section 15.8), with equation :*

$$n\mathcal{K}\left(P,U\right) \doteq ux\left(r\,y^2 + qz^2\right) + vy\left(p\,z^2 + r\,x^2\right) + wz\left(q\,x^2 + p\,y^2\right) + k\,xyz \tag{15.3}$$

Proof. Direct inspection from $\mathcal{K}\left(X_P^*\right) = \lambda\mathcal{K}\left(X\right)$. It can be seen that terms like x^3 are to be avoided, and that terms like xy^2 and xz^2 are to be paired. □

$p\mathcal{K}$	name		F	U,U^*	E,E^*	D,D^*	some other points on the cubic
$p\mathcal{K}_A$	vertex	15.3.3	1	A			
ZU(1)		15.3.4	1	X(1)			
E_1, E_3	hidden	15.3.9	1	Ω			
K002	Thomson	15.3.17	1	2,6	3,4	9,57	223, 282, 1073, 1249
K003	McKay	15.3.10	1	3,4	1075,?	1745,3362	
K006			1	4,3	155,254	46,90	371, 372, 485, 486, 487, 488
K005			1	5,54	2120,2121	3460,3461	3, 4, 17, 18, 61, 62, 195, 627, 628
K004	Darboux	15.4	1	20,64	2130,2131	3182,3347	3, 4, 40, 84, 1490, 1498
K001	Neuberg	15.3.11	1	30,74	2132,2133	3464,?	3, 4, 13, 14, 15, 16, 399, 484, ...
			1	98,511	?,?	1756,?	1687, 1688, 2009, 2010
K035			1	99,512	39,83	1019,1018	1379, 1380
			1	100,513	1,1	513,100	1381, 1382
			1	110,523	5,54	?,?	1113, 1114
K020			1	384,695	?,?	?,?	3,4,32,39,76,83
K021			1	512,99	2142,2143	?,?	
	shortest	15.3.13	1	523,110			
K007	Lucas	15.4	2	X(69)			
K170		15.2.5	2	X(4)			
K155	EAC2	15.5.2	$\sqrt{31}$	238			
K060		15.6	$\sqrt{1989}$	265			
$n\mathcal{K}$			F	root			
Δ		15.3.5	1				
circle		15.3.7	1				
K137			1	513	$Z^+(X_1 X_6)$		1, 44, 88, 239, 241, 292, 294, 1931
???			1	649	$Z^+(X_1 X_2)$		1, 238, 291, 899, 2107
K040	Pelletier		1	650	$Z^+(X_1 X_3)$		1,105,243,296,518,1155,1156, 2651, 2652
K018	Brocar2	15.13	1	523	$Z^+(X_3 X_6)$		2, 6, 13, 14, 15, 16, 111, 368, 524
K010	Simson	15.10	2		$c\mathcal{K}\,(\#X_2,\, X_{69})$		2, X(2394) upto X(2419)
K162		15.10.4	6		$c\mathcal{K}\,(\#X_6,\, X_3)$		6, X(2420) upto X(2445)

$F=\sqrt{P}$ (central fixed point), U, U_P^*, $E = cevadiv\,(U,U^*)$, E^*, $D = cevadiv(U,\sqrt{P})$

Table 15.1: Some well-known cubics

15.2 Pivotal isocubics pK(P,U)

Remark 15.2.1. The isocubic property is better stated as follows : When X is on the sidelines of triangle ABC, then $X' = X_F^{\#}$ is undefined (geometrically) and formula gives a vertex. Otherwise, X' belongs to $p\mathcal{K}\,(\#F,U)$ if and only if X belongs to $p\mathcal{K}\,(\#F,U)$.

Remark 15.2.2. The cevadiv property is better stated as follows : When X is on the sidelines of the cevian triangle of U, $X' = cevadiv\,(U,X)$ is undefined (geometrically) and formula gives $0:0:0$. Otherwise, X' belongs to $p\mathcal{K}\,(\#F,U)$ if and only if X belongs to $p\mathcal{K}\,(\#F,U)$.

Proposition 15.2.3. *The 20 points property. The* $p\mathcal{K}\,(\#F,U)$ *cubic goes through* ABC, $U A_U B_U C_U$, $F_0 F_A F_B F_C$ *(cf Theorem 15.1.3) and also through cevadiv* $\left(U,\, U_F^{\#}\right)$, *the four cevadiv* $(U,\, F_j)$ *and their isoconjugates.*

Proof. Direct examination. □

Example 15.2.4. When U is one of the fixed points of the isoconjugacy (i.e. $P = U *_b U$), the $p\mathcal{K}(P, U)$ cubic degenerates into the lines through the remaining three fixed points.

Example 15.2.5. K170 est $p\mathcal{K}(X_2, X_4)$. Equation $\sum x\left(y^2 - z^2\right)/S_a = 0$. On Figure 15.1, one can see the following alignments (general properties, applicable to any $p\mathcal{K}$) :

1. Fixed points : F_0, F_a, A are collinear, and cyclically for the other fixed points and the other vertices ;
2. From U : U, X, $X_F^{\#}$ are collinear (e.g. E and E^* are aligned with U). Therefore, each line from U to a fixed point is tangent to the cubic at this fixed point ; in the same vein, point $U_b = UB \cap AC$ is on the cubic and viewing B as $(U_b)_F^{\#}$ makes sense, but not viewing U_b as $B_F^{\#}$ (this object would be "quite all points on line AC").
3. From $U_F^{\#}$: $U_F^{\#}$, X, *cevadiv* (U, X) are collinear (e.g. F and D are aligned with $U_F^{\#}$). Therefore, each line from $U_F^{\#}$ to a vertex or to U is tangent to the cubic at this point.

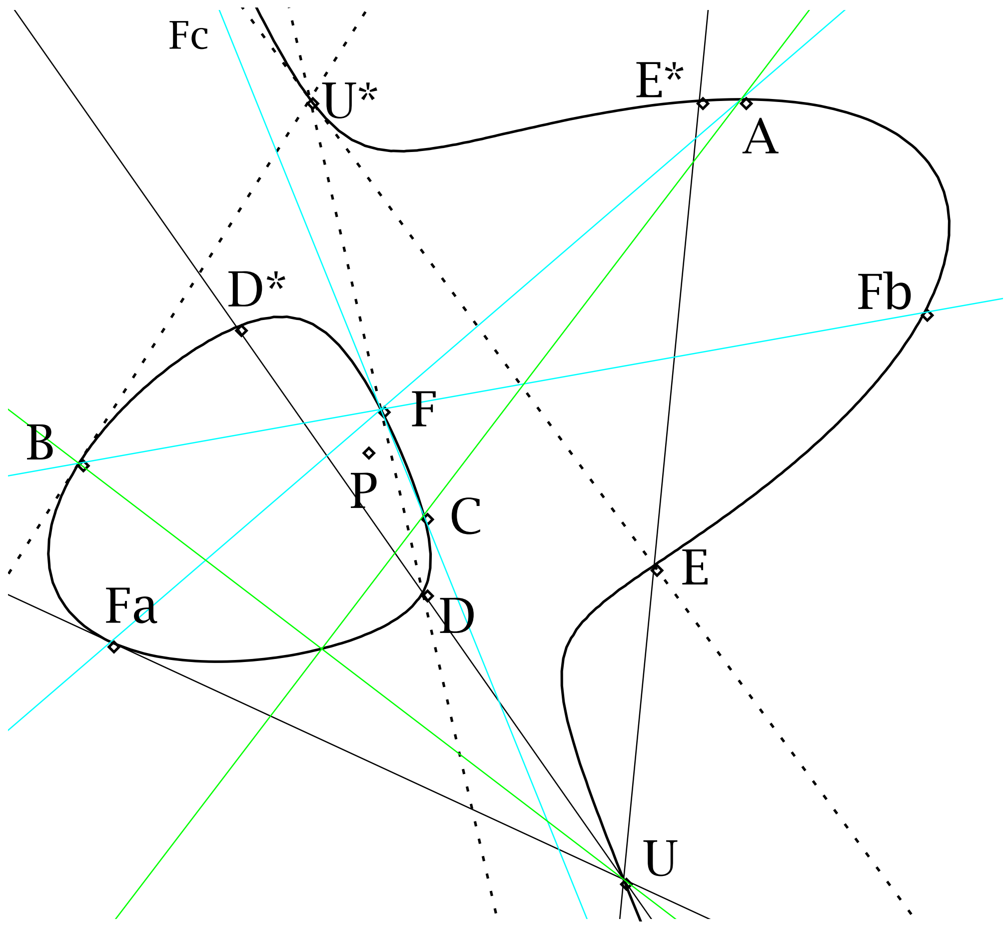


Figure 15.1: pK(2,4)

Proposition 15.2.6. *The $PK(X)$ point is the intersection of the trilinear polars of points X and X_P^* (Bernard Gibert, 2003/10/1). Using barycentrics and $P = p : q : r = f^2 : g^2 : h^2$, we have :*

$$\begin{aligned} PK_P(X) &= p\,x\left(r\,y^2 - q\,z^2\right) : q\,y\left(p\,z^2 - r\,x^2\right) : rz\left(q\,x^2 - p\,y^2\right) \\ &= crossdiff_F\left(X,\, X_P^*\right) \end{aligned}$$

Proof. Direct computation (see 18.3 for the definition of *crosdiff*). □

Example 15.2.7. Using F=X(1), i.e. P=X(6), we have PK(X(I)) = X(J) for these (I,J):

I	2	3	4	5	6	9	19	31	44	54	57	63
J	512	647	647	2081	512	663	810	2084	3251	2081	663	810

Proposition 15.2.8. *When $PK_P(X)$ belongs to the tripolar line of U_P^*, then point X belongs to the cubic $p\mathcal{K}(P,U)$.*

Proof. Direct computation. In Kimberling (1998, p. 240) the corresponding cubic is noted $Z(U, Y)$ where $P = Y *_b X_6$. □

Proposition 15.2.9. *When point $U = u : v : w$ is at infinity, the $p\mathcal{K}(P,U)$ cubic can be rewritten as :*

$$\left(\frac{p}{x}+\frac{q}{y}+\frac{r}{z}\right)(x\rho+y\sigma+z\tau)-(x+y+z)\left(\frac{a^2\rho}{x}+\frac{b^2\sigma}{y}+\frac{c^2\tau}{z}\right)=0$$

where $[\rho,\sigma,\tau]$ is any line whose direction is U. In other words $u=\sigma-\tau$, etc.

Proof. Direct examination. We can check that $p\mathcal{K}(P,U)$ contains the intersections of both conics —vertices A, B, C and U_P^*—, the points at infinity of $\mathcal{C}_{cir}(P)$, the intersections of line $[\rho,\sigma,\tau]$ with the associated conic, and U itself ! □

15.3 pK isocubics relative to pole X(6)

Definition 15.3.1. A Kimberling ZU cubic is a $p\mathcal{K}(X_6, U)$, giving a special place to isogonal conjugacy. Some examples are given in Table 15.1.

Proposition 15.3.2. *Only 8 ZU cubics have a reflection center : the Darboux cubic (center= X_3), the four degenerate cubics that are union of the three bisectors through an inexcenter, and three other (Maple length = 135712 avec RootOf, [4948, 5345, 4215] avec alias).*

15.3.1 $ABCIJKL$ cubics: the Lubin(2) point of view

Proposition 15.3.3. ***Cubics PKA.*** *The set of all cubics that go through points $ABCIJKL$ is a projective space. Its dimension is* 3. *A generating familly is made of the three cubics $p\mathcal{K}_A = (BC)\cup(AI)\cup(KL)$*, etc. *Pivot of $p\mathcal{K}_A$ is A. Its factored equation in Lubin(2) is :*

$$p\mathcal{K}_A=\det\begin{pmatrix}\beta^2 & \gamma^2 & \mathbf{Z}\\ 1 & 1 & T\\ \dfrac{1}{\beta^2} & \dfrac{1}{\gamma^2} & \overline{\mathcal{Z}}\end{pmatrix}\det\begin{pmatrix}\alpha^2 & \alpha\beta+\beta\gamma+\gamma\alpha & \mathbf{Z}\\ 1 & -1 & T\\ \dfrac{1}{\alpha^2} & \dfrac{\alpha+\beta+\gamma}{\alpha\beta\gamma} & \overline{\mathcal{Z}}\end{pmatrix}\det\begin{pmatrix}\alpha\beta+\beta\gamma-\gamma\alpha & \beta\gamma+\gamma\alpha-\alpha\beta & \mathbf{Z}\\ 1 & 1 & T\\ \dfrac{\alpha-\beta+\gamma}{\alpha\beta\gamma} & \dfrac{\alpha+\beta-\gamma}{\alpha\beta\gamma} & \overline{\mathcal{Z}}\end{pmatrix}$$

while its (not factorizable) equation in Lubin(1) is $\det(X, X_P^*, A) = 0$ *where X_P^* is given by the isogonal conjugacy formula.*

Proof. Equation of $p\mathcal{K}(F,U)$ is $\det\left(X,\, X_F^{\#},\, U\right)=0$, leading to dimension 3, and allowing to check that pivot of $p\mathcal{K}_A$ is the vertex A. After that, we can go back to Lubin(1) since isogonal conjugacy doent require to identify which is the incenter among the inexcenters. □

Example 15.3.4. The Kimberling $Z(X(1))$ cubic, i.e. $(IJ)\cup(IK)\cup(IL)$, is obtained as :

$$\frac{(\beta+\gamma)}{(\beta-\gamma)}\,p\mathcal{K}_A+\frac{(\gamma+\alpha)}{(\gamma-\alpha)}\,p\mathcal{K}_B+\frac{(\alpha+\beta)}{(\alpha-\beta)}\,p\mathcal{K}_C$$

Definition 15.3.5. The **triangular cubic** is the union of the three sidelines: $n\mathcal{K}_0 = (BC)\cup(CA)\cup(AB)$. Its normalized equation is defined by :

$$n\mathcal{K}_0(X)=\det(A,B,X)\,\det(A,X,C)\,\det(X,B,C)\div\det(ABC)$$

Theorem 15.3.6. *When substituting the isogonal formulae* (14.3) *into the equation of a #X(1) pivotal cubic, we have the following equalities :*

$$\begin{aligned} p\mathcal{K}\left(isog\left(X\right)\right) &= +p\mathcal{K}\left(X\right)\times n\mathcal{K}_0\left(X\right)\\ n\mathcal{K}\left(isog\left(X\right)\right) &= -n\mathcal{K}\left(X\right)\times n\mathcal{K}_0\left(X\right)\end{aligned}$$

that are exact identities, not up to a proportionality factor. Therefore, the set of the isocubics splits into two projective subspaces, the $p\mathcal{K}$ one (dimension 3) and the $n\mathcal{K}$ one (dimension 4).

Definition 15.3.7. Cubic "circumcircle union infinity", i.e. $\mathbf{T}\left(\mathbf{Z}\overline{\mathcal{Z}}-\mathbf{T}^2\right)$ is a $n\mathcal{K}$.

Proposition 15.3.8. *The Kiepert RH construction (see Proposition 10.18.1 for more details) can be summarized as follows. Let $K=\cot\phi$ be a fixed real and define circulary the points $A'B'C'$ by :*

$$\cot\left(\overbrace{BC,\,BA'}\right)-K=0\;;\;\cot\left(\overbrace{BC,\,CA'}\right)+K=0$$

These points are cocubics with $ABCIJKL$. To prove this result, we obtain :

$$A'\simeq\beta+\gamma+i\left(\gamma-\beta\right)K:2:\left(\beta+\gamma+i\left(\gamma-\beta\right)K\right)\div\beta\gamma$$

Then substitute A' and B' into $\sum_3 K_j p\mathcal{K}_j=0$. This system has non zero solutions, proving the result. Moreover, the obtained cubic can be rewritten using the Lubin(1) basis and we have :

$$\begin{aligned} p\mathcal{K}_{(K)} &= p\mathcal{K}_{Thomson}+K^2\,p\mathcal{K}_{Darboux}\\ &= \left(K^2+1\right)p\mathcal{K}_{Neuberg}+\left(3-K^2\right)p\mathcal{K}_{McKay}\end{aligned}$$

Equations of the named cubics are given below. We can see that $p\mathcal{K}_{(K)}$ contains also the circumcenter $X(3)$ (the $\mathbf{T}^3$ coefficient), the orthocenter (isogonality) and the $A'B'C'$ points related with the other orientation ($p\mathcal{K}_{(K)}$ depends only on K^2).

15.3.2 A more handy basis

Proposition 15.3.9. The hidden IJKL cubics. *The following cubics are conjugate of each other, but not self conjugate (hidden curves). Their nine visible intersections are $ABCIJKL$ (stable by isogonal conjugacy) and the two umbilics (stable as a pair) :*

$$\begin{aligned} E_1 &= \mathbf{Z}^2\overline{\mathcal{Z}}+\sigma_3 T\overline{\mathcal{Z}}^2-2T^2\mathbf{Z}-\sigma_2 T^2\overline{\mathcal{Z}}+T^3\sigma_1\\ E_3 &= \mathbf{Z}\overline{\mathcal{Z}}^2+\frac{1}{\sigma_3}T\mathbf{Z}^2-\frac{\sigma_1}{\sigma_3}T^2\mathbf{Z}-2T^2\overline{\mathcal{Z}}+\frac{\sigma_2}{\sigma_3}T^3\end{aligned}$$

Both curves are $p\mathcal{K}$ cubics and pivot of E_1 is the umbilic $0:0:1$ while umbilic $1:0:0$ is the pivot of E_2.

Proof. Start from (14.3) that gives $isog\left(X\right)$ when $X\in\mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^3\right)$ and compute :

$$E=\left(\mathbf{T}^2-\mathbf{Z}\overline{\mathcal{Z}}\right)X-\mathbf{T}\,isog\left(X\right)$$

Quantities E_1,E_3 are the $\mathbf{Z},\overline{\mathcal{Z}}$ components of E while factors have been chosen to ensure $E_2=0$. Column E vanishes at an umbilic since the factors are zero. At a vertex, the circle vanishes and $isog\left(X\right)$ is $0:0:0$. At a fixed point, this column is proportional to X but E_2 is ever 0 and the X aren't at infinity. This result has been proved in the Lubin(1) space, without specifying who is the incenter among the three inexcenters. Obviously, is can be checked in Lubin(2) by substitution or even by factoring the $\mathbf{T}$ resultant of both equations. Thereafter, pivots are obtained by identification. □

Proposition 15.3.10. ***K003, the McKay Cubic**, $p\mathcal{K}(6,3)$. Expression $\left(\mathbf{Z}\,E_2-\overline{\mathcal{Z}}\,E_1\right)/T$ gives a visible cubic, whose equation is :*

$$p\mathcal{K}_{McKay}=\frac{1}{\sigma_3}\mathbf{Z}^3-\sigma_3\overline{\mathcal{Z}}^3-\frac{\sigma_1}{\sigma_3}T\mathbf{Z}^2+\sigma_2 T\overline{\mathcal{Z}}^2+\frac{\sigma_2}{\sigma_3}T^2\mathbf{Z}-\sigma_1 T^2\overline{\mathcal{Z}}$$

K003 goes through the orthocenter. Its points at infinity are $\Theta : 1/\Theta : 0$ *where* $\Theta^2 = \sqrt[3]{\sigma_3^2}$. *These points are the directions of the Morley triangles. We obtain the asymptotes as the gradient at the corresponding points. We obtain :*

$$\left[\ 3\,\frac{\Theta^2}{\sigma_3}\quad ;\quad \frac{\sigma_2}{\Theta^2} - \frac{\sigma_1\,\Theta^2}{\sigma_3}\quad ;\quad -3\,\frac{\sigma_3}{\Theta^2}\ \right]$$

and the other two are obtaines by susbstituting Θ *by* $j\,\Theta$ *or* $j^2\,\Theta$. *By supersymmetry, the three asymptotes concur, at the gravity center. The pivot is X(3).*

Proof. Everything is straightforward, except from the pivot. It can be obtained as the intersection of two well chosen lines PP^*, for example for two of the points at infinity. □

Proposition 15.3.11. ***K001, the Neuberg cubic***, $p\mathcal{K}\,(6,30)$. *Quantity* $(\sigma_2/\sigma_3)\,E_1 - \sigma_1 E_2$ *provides another visible cubic. Its equation is :*

$$p\mathcal{K}_{Neuberg} = \frac{\sigma_2}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}} - \sigma_1\,\mathbf{Z}\overline{\mathcal{Z}}^2 - \frac{\sigma_1}{\sigma_3}\,T\,\mathbf{Z}^2 + \sigma_2\,T\overline{\mathcal{Z}}^2 + \frac{\sigma_1^2 - 2\,\sigma_2}{\sigma_3}\,T^2\mathbf{Z} + \frac{2\,\sigma_1\,\sigma_3 - \sigma_2^2}{\sigma_3}\,T^2\overline{\mathcal{Z}}$$

This is a circular curve. The third point at infinity is $\sigma_1\sigma_3 : 0 : \sigma_2$, *i.e. X(30), the direction of the Euler line. This is also the pivot. The asymptotes are :*

$$\left[\ 0\quad -\sigma_1\quad \sigma_2\ \right],\ \left[\ -\sigma_1\quad \sigma_2\quad 0\ \right],\ \left[\ \sigma_2^2\sigma_1\quad ;\quad -\sigma_1^3\sigma_3 + \sigma_2^3\quad ;\quad -\sigma_2\,\sigma_1^2\,\sigma_3\ \right]$$

Intersecting the first two asymptotes, we obtain a singular focus at $\sigma_2^2 : \sigma_1\,\sigma_2 : \sigma_1^2$, *i.e. point X(110). Moreover, the cubic goes through X(3) and therefore through X(4).*

Proof. X(30) is the pivot because it is the only visible common point to the curve and line PP^* when P is an umbilic. An asymptote is the gradient evaluated at the corresponding point at infinity. □

Corollary 15.3.12. *The Neuberg cubic is the locus of points* X *such that the isogonal line* $X\,X^*$ *is parallel to the Euler line. See (Gibert, 2004-2010, 2005). Its (barycentric) equation is :*

$$\sum_{cyclic} x\left(y^2c^2 - z^2b^2\right)\left(2\,a^4 - \left(b^2 + c^2\right)a^2 - \left(b^2 - c^2\right)^2\right) = 0 \tag{15.4}$$

and can be rewritten as :

$$\left(\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}\right)(x\rho + y\sigma + z\tau) - (x + y + z)\left(\frac{a^2\rho}{x} + \frac{b^2\sigma}{y} + \frac{c^2\tau}{z}\right) = 0$$

where $[\rho, \sigma, \tau]$ *is the Euler line –remember: tripole = X(648). In other words, K001=circumcircle×Euler - infinity×Jerabek.*

Proof. See Proposition 15.2.9. We can check that K001 contains the intersections of Γ and Jerabek —vertices A, B, C and X(74)—, the points at infinity of the circumcircle —the umbilics—, the intersections of Euler line and Jerabek hyperbola —X(3) and X(4)— and X(30) itself. □

Proposition 15.3.13. ***Shortest cubic***, $p\mathcal{K}\,(6,523)$. *Quantity* $(1/\sigma_1)\,E_1 + (\sigma_3/\sigma_2)\,E_2$ *provides another visible cubic. Its equation is :*

$$p\mathcal{K}_{shortest} = \frac{1}{\sigma_1}\,\mathbf{Z}^2\overline{\mathcal{Z}} + \frac{\sigma_3}{\sigma_2}\,\mathbf{Z}\overline{\mathcal{Z}}^2 + \frac{1}{\sigma_2}\,\mathbf{Z}^2T + \frac{\sigma_3}{\sigma_1}\,T\overline{\mathcal{Z}}^2 - \frac{\sigma_1^2 + 2\,\sigma_2}{\sigma_2\,\sigma_1}\,T^2\mathbf{Z} - \frac{\sigma_2^2 + 2\,\sigma_1\,\sigma_3}{\sigma_2\,\sigma_1}\,T^2\overline{\mathcal{Z}} + 2\,T^3$$

The third point at infinity is $-\sigma_1\sigma_3 : 0 : \sigma_2$, *i.e. X(523), the orthodir of the Euler line. This is also the pivot. The umbilical asymptotes concur at X(110), giving a singular focus. Barycentric equation of this curve is ;*

$$\left(b^2 - c^2\right)x\left(b^2z^2 - c^2y^2\right) + \left(c^2 - a^2\right)y\left(c^2x^2 - a^2z^2\right) + \left(a^2 - b^2\right)z\left(a^2y^2 - b^2x^2\right)$$

The ETC databasis doesn't contains other known points. We have the eight points: the 4 inexcenters, X(110), X(523) and both umbilics.

Remark 15.3.14. The "shortest cubic" has been introduced as the circular $p\mathcal{K}$ cubic whose expression requires the shortest number of characters. Its main property is to provide an handy basis when used together with the McKay and the Neuberg cubics. The shortest cubic contains no ETC points apart X(1), X(523) and X(110).

Proposition 15.3.15. ***Resulting pencils***. *We have the following pencils :*

1. *The pencil generated by $p\mathcal{K}_{Neuberg}$ and $p\mathcal{K}_{shortest}$ is the set of all the circular $p\mathcal{K}$ cubics. Their pivots are at infinity.*
2. *The pencil generated by $p\mathcal{K}_{McKay}$ et $p\mathcal{K}_{Neuberg}$ is the set of the $p\mathcal{K}$ cubics that goes through the X(3), X(4) pair. Their pivots are on the Euler line.*
3. *The pencil generated by $p\mathcal{K}_{McKay}$ et $p\mathcal{K}_{shortest}$ is the set of $p\mathcal{K}$-cubics whose pivots are on the line X(3), X(523), i.e. the line through X(3) and perpendicular to the Euler line. The common points of these cubics are on Δ (horrible formula, with a 24th degree radicand).*

Exercise 15.3.16. Does it exist a cubic XXX such that :

1. XXX, K001, K003 provide a basis of the ZU cubics space
2. XXX contains "many" ETC points
3. Pencils (XXX,K001) and (XXX,K003) contain "many" known cubics
4. Lubin equation of XXX remains practicable

Proposition 15.3.17. K002, the Thomson cubic, $p\mathcal{K}\,(6,2)$ *is the locus of points X whose trilinear polar is parallel to their polar line in the circumcircle. It is a $p\mathcal{K}$ cubic with pole=X(6) and pivot=X(2). It is the $K=0$ cubic of the Kieper RH construct. Its equation is :*

$$p\mathcal{K}_{Thomson}=\frac{3}{\sigma_3}\,\mathbf{Z}^3+\frac{\sigma_2}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}}-\sigma_1\,\mathbf{Z}\overline{\mathcal{Z}}^2-3\,\sigma_3\,\overline{\mathcal{Z}}^3-\frac{4\sigma_1}{\sigma_3}\,T\,\mathbf{Z}^2+4\,\sigma_2\,T\overline{\mathcal{Z}}^2+\frac{\sigma_2+\sigma_1^2}{\sigma_3}\,T^2\mathbf{Z}-\frac{\sigma_2^2+\sigma_1\,\sigma_3}{\sigma_3}\,T^2\overline{\mathcal{Z}}$$

15.4 Darboux and Lucas cubics aka K004 and K007

In this section, $P\in Darboux$ and $U\in Lucas$ while pole and pivot are noted otherwise.

Definition 15.4.1. The Darboux cubic K004 is the locus of point P such that the pedal triangle of P is the Cevian triangle of some other point U, while the Lucas cubic K007 is the locus of point U such that the cevian triangle of U is the pedal triangle of some other point P. .

Proposition 15.4.2. *Darboux cubic is a $p\mathcal{K}$ cubic, with X(6) as pole and X(20) as pivot. X(20) is the de Longchamps point. Lucas cubic is a $p\mathcal{K}$ cubic, with X(2) as pole and X(69) as pivot. X(69) is the anticomplement of X(6). Their equations are :*

$$\begin{aligned}\det\left(X_{20},\,P,\,isog\,(P)\right)&=0 &(15.5)\\ \det\left(X_{69},\,U,\,isot\,(U)\right)&=0 &(15.6)\end{aligned}$$

Moreover, K004 has a reflection center at X(3), the circumcenter. Using Morley affixes and expanding, we have :

$$p\mathcal{K}_{Darboux}=-\frac{\mathbf{Z}^3}{\sigma_3}+\frac{\sigma_2}{\sigma_3}\,\mathbf{Z}^2\overline{\mathcal{Z}}-\sigma_1\,\mathbf{Z}\overline{\mathcal{Z}}^2+\sigma_3\,\overline{\mathcal{Z}}^3+\frac{\sigma_1^2-3\,\sigma_2}{\sigma_3}\,T^2\mathbf{Z}+\frac{3\,\sigma_1\,\sigma_3-\sigma_2^2}{\sigma_3}\,T^2\overline{\mathcal{Z}}$$

Proof. Straightforward from (6.1) and (3.4). □

Proposition 15.4.3. *The correspondence between $P\in Darboux$ and $U\in Lucas$ is described by :*

$$\psi\begin{pmatrix}p\\ q\\ r\end{pmatrix}\simeq\begin{pmatrix}(ar+cp\cos B)\,(bp+aq\cos C)\\ (aq+bp\cos C)\,(br+cq\cos A)\\ (br+cq\cos A)\,(ar+cp\cos B)\end{pmatrix}\tag{15.7}$$

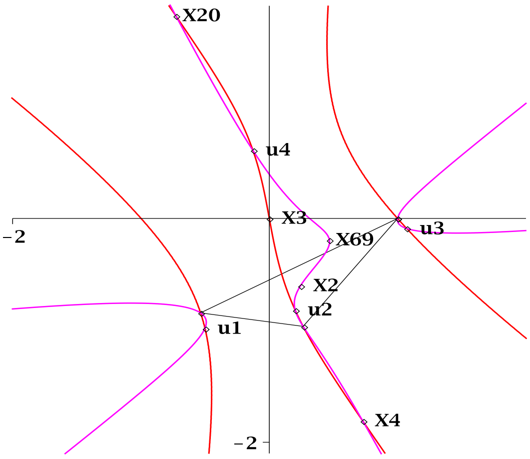


Figure 15.2: The Darboux and Lucas cubics

$$\psi^{-1} \simeq \left(\begin{array}{c} -\dfrac{a^3 b\cos C}{u} + \dfrac{b^2a^2}{v} + \dfrac{a^2bc\cos A}{w} \\ \dfrac{b^2a^2}{u} - \dfrac{ab^3\cos C}{v} + \dfrac{ab^2c\cos B}{w} \\ \dfrac{\alpha}{u} + \dfrac{\beta}{v} - \dfrac{\gamma}{w} - \dfrac{w}{4\,u\,v}\left(a^4 - 2\,b^2a^2 - 2\,a^2c^2 + c^4 - 2\,b^2c^2 + b^4\right) \end{array} \right) \tag{15.8}$$
$$\textit{where}\quad \alpha = a^2bc\,\cos C\,\cos B,\ \beta = ab^2c\,\cos C\,\cos A,\ \gamma = abc^2\,\cos A\,\cos B$$

Proof. Transformation ψ as described in (15.7) and (15.8) is certainly not a central transformation when applied to the whole plane. But this transformation is central when restricted to the Darboux cubic and nothing else matters. On the other hand, a more symmetrical formula will involve cubic roots and, for this reason, will not be efficiently used by formal computing tools. □

In Kimberling (2002a), the following chains are emphasized. When P is on *Darboux* then $isog\,(P)$ too (the cyclopedal property), so that $\psi\,(P)$ and $(\psi\circ isog)\,(P)$ belong to *Lucas*. In this figure are involved two inscribed triangles that share the same circumcircle, so that

$$(\psi\circ isog)\,(P) = (cyclocev\circ\psi)\,(P)$$

When $P = X_3$, the corresponding circumcenter is X_5; when $P = X_{40}$, this center is X_{1158}. Now, from (15.6), *isot* is an involution of *Lucas*. Therefore, $\psi^{-1}\circ isot\circ\psi$ is an involution of *Darboux*. On the other hand, the symmetry centered at X_3, i.e. the transformation :

$$sym3\,(U) \simeq 2\,[\alpha,\beta,\gamma]\,/(\alpha+\beta+\chi) - [u,v,w]\,/\,(u+v+w) \tag{15.9}$$

where $X_3 = \alpha : \beta : \gamma$ is a natural involution of the Darboux cubic. It is natural to check if they are equal. This is straightforward when using (15.7) and (15.8).

$$\begin{array}{ccccccccccccc}
\boxed{X_1} & \xrightarrow{sym3} & X_{40} & \xrightarrow{isog} & X_{84} & \xrightarrow{sym3} & X_{1490} & \xrightarrow{isog} & X_{3345} & \xrightarrow{sym3} & X_{3182} & \cdots \\
\downarrow\psi & & \downarrow\psi & & \downarrow\psi & & \downarrow\psi & & \downarrow\psi & & \downarrow\psi & \\
\boxed{X_7} & \xrightarrow{isot} & X_8 & \xrightarrow{cycl} & X_{189} & \xrightarrow{isot} & X_{329} & \xrightarrow{cycl} & X_{1034} & \xrightarrow{isot} & X_{???} & \cdots \\
\boxed{X_3} & \xrightarrow{isog} & X_4 & \xrightarrow{sym3} & X_{20} & \xrightarrow{isog} & X_{64} & \xrightarrow{sym3} & X_{1498} & \xrightarrow{isog} & X_{3346} & \cdots \\
\downarrow\psi & & \downarrow\psi & & \downarrow\psi & & \downarrow\psi & & \downarrow\psi & & \downarrow\psi & \\
\boxed{X_2} & \xrightarrow{cycl} & X_4 & \xrightarrow{isot} & X_{69} & \xrightarrow{cycl} & X_{253} & \xrightarrow{isot} & X_{20} & \xrightarrow{cycl} & X_{1032} & \cdots
\end{array}$$

Since X_1 is fixed by *isog* and X_3 by *sym*3, these chains are unidirectional.

Claim 15.4.4. Let $Q = (sym3\circ isog\circ sym3\circ isog\circ sym3)\,P$. When $P\in Darboux$, then $cevamul\,(P,\,Q) = X_{20}$ (for all known points, and randomly). When P is on the branch of X_3, so is Q (obvious). When P is not on *Darboux*, ??? In any case, a simple division by polynomial (15.5) isn't sufficient.

R	A'	A''	B''
$a^{3/2}$	A	$-a^{3/2}$	$b^{3/2}$
$X(2)$	A	$-a^2$	bc
$X(238)$	$-a^2 : bc :$	$-abc$	b^3
$X(31)$	A	$-abc$	b^3
$X(1)$	A	$-a^2(a+b+c)$	$b(bc+ca+ab)$
$X(238)$	upright	$-a(bc+ca+ab)$	$b^2(a+b+c)$
$X(6)$	A	$-a(bc+ca+ab)$	$b^2(a+b+c)$
$X(6)$	$-a^2 : bc :$	$a^2(a+b+c)$	$b\left(a^2+b^2-c(a+b)\right)$
$X(1)$	$-abc : b^3 :$	$a^2(bc+ca+ab)$	$(c-b)a^2b+b^3(c-a)$
$X(31)$	$-a^2 : bc :$	$2a^2\left(b^2+ca\right)\left(c^2+ab\right)$ $b\left(b^2+ca\right)\left(a^3+b^3-c^3-abc\right)$	
$X(1)$	$-a^2 : bc :$	$a(a+b)(c+a)\left(a^2+b^2+c^2+bc+ca+ab\right)$ $b^2(c+a)\left(a^2+b^2+ab-c(a+b+c)\right)$	
$X(2)$	$-abc : b^3 :$	$2a^3bc\left(b^2+ca\right)\left(c^2+ab\right)$ $\left(b^2+ca\right)\left(b^4c^3+a^3bc^3-a^3b^4-b^3a^2c^2\right)$	
$X(6)$	$-abc : b^3 :$	$a(a+b)(c+a)\left(\left(c^2+bc+b^2\right)a^2+bc(b+c)a+b^2c^2\right)$ $b^2(c+a)\left(\left(c^2-bc-b^2\right)a^2+abc(-b+c)+b^2c^2\right)$	

Table 15.2: Some cubic shadows on EAC2

15.5 Equal areas (second) cevian cubic aka K155

Definition 15.5.1. Cubic shadow. Triangle centers on a cubic $\mathcal{K}$ yield non-central points on the cubic; e.g., if Q_1 and Q_2 are on $\mathcal{K}$, then the line Q_1Q_2 meets $\mathcal{K}$ in a "third" point, $L(Q_1,Q_2)$, possibly Q_1 or Q_2. If $A'B'C'$ is a central triangle (cf Section 2.2), R a triangle center, $A''=L(R,A')$ and cyclically, then triangle $A''B''C''$ is a central triangle on $\mathcal{K}$.

Definition 15.5.2. Cubic EAC2, the equal areas (second) cevian cubic is $K155$ in Gibert (2004-2010). This cubic is $p\mathcal{K}(X31, X238)$, i.e self-isoconjugate wrt $P=X_{31}=a^3:b^3:c^3$ and pivotal wrt $U=X_{238}=a^3-abc:b^3-abc:c^3-abc$.

Proposition 15.5.3. *It happens that $P\in EAC2$. When a point Q is on EAC2, its isoconjugate Q_P^* aka $X31 \div_b Q$ is on EAC2 too. In the following table, for each (I,J), the centers X(I) and X(J) are on EAC2 and are an isoconjugate pair. Each pair is collinear with the pivot X(238).*

$$\begin{bmatrix} 1 & 2 & 105 & 238 & 365 & 1423 & 1931 \\ 6 & 31 & 672 & 292 & 365 & 2053 & 2054 \end{bmatrix}$$

$$\begin{bmatrix} 2106 & 2108 & 2110 & 2112 & 2114 & 2116 & 2118 & 2144 & 2146 \\ 2107 & 2109 & 2111 & 2113 & 2115 & 2117 & 2119 & 2145 & 2147 \end{bmatrix}$$

Table 15.2 gives some cubic shadows on EAC2. Column 1 gives the perspector $R\in\mathcal{K}$. Column 2 gives $A'\in\mathcal{K}$, the A vertex of the original triangle. Columns 3 and 4 give $A''\in\mathcal{K}$, the A vertex of the shadow triangle. When expressions are growing, these coordinates are given in two rows. For example, in row 1, the perspector is the centroid, the original triangle is ABC itself and $A''=-a^2:bc:bc$, the third coordinate being obtained by swapping b and c. Points X_2, $A'=A$ and A'' are collinear. Obviously, $(A'')_P^*$, $(B'')_P^*$, $(C'')_P^*$, is another central triangle inscribed in $\mathcal{K}$.

15.6 The cubic K060

Proposition 15.6.1. *Let A', B', C' be the reflections of a point M into the sidelines BC, CA, AB. When triangle $A'B'C'$ is perspective with ABC, point M lies on the Neuberg cubic K001, while the resulting perspector N lies on another cubic (K060).*

Proof. The matrix of the reflection into the sideline BC is :

$$\boxed{\sigma_A} \simeq \left(\begin{array}{ccc} -a^2 & 0 & 0 \\ a^2+b^2-c^2 & a^2 & 0 \\ a^2-b^2+c^2 & 0 & a^2 \end{array}\right)$$

Start from $M = p:q:r$. Compute $A' = \sigma_A(M)$, etc and obtain :

$$A'B'C' \simeq \left(\begin{array}{ccc} -pa^2 & \left(a^2+b^2-c^2\right)q+pb^2 & \left(a^2-b^2+c^2\right)r+pc^2 \\ \left(a^2+b^2-c^2\right)p+a^2q & -qb^2 & \left(b^2+c^2-a^2\right)r+qc^2 \\ \left(a^2-b^2+c^2\right)p+a^2r & \left(b^2+c^2-a^2\right)q+b^2r & -rc^2 \end{array}\right)$$

Then $\det(AA', BB', CC')$ is computed and identified with $K001$. Now, start from $N = u:v:w$. Compute $\delta_A = \sigma_A(AN) = (A \wedge N) \cdot \boxed{\sigma_A}^{-1}$, etc and obtain :

$$\left(\begin{array}{c} \delta_A \\ \delta_B \\ \delta_C \end{array}\right) \simeq \left(\begin{array}{ccc} 2\,vS_b - 2\,wS_c & -wa^2 & va^2 \\ wb^2 & 2\,wS_c - 2\,uS_a & -ub^2 \\ -vc^2 & uc^2 & 2\,uS_a - 2\,vS_b \end{array}\right)$$

Then $\det(\delta_A, \delta_B, \delta_C)$ is computed, and we obtain yet another $p\mathcal{K}$ cubic, defined as K060. □

Proposition 15.6.2. K060, the $p\mathcal{K}\,(1989, 265)$ **cubic**. *The pole P, the pivot U and the Morley equations of this cubic are respectively :*

$$P = \frac{1}{b^2c^2 - 4\,S_a^2} \text{ etc} \quad ; \quad z_P = \frac{\sigma_1^3\sigma_2^2 - \sigma_1\sigma_2^3 - \left(4\,\sigma_1^4 - 9\,\sigma_1^2\sigma_2 + 9\,\sigma_2^2\right)\sigma_3}{3\,\sigma_1^2\sigma_2^2 - 6\,\sigma_2^3 + \left(9\,\sigma_1\sigma_2 - 6\sigma_1^3\right)\sigma_3}$$

$$U = \frac{S_a}{b^2c^2 - 4\,S_a^2} : \frac{S_b}{a^2c^2 - 4\,S_b^2} : \frac{S_c}{a^2b^2 - 4\,S_c^2} \quad ; \quad z_U = \frac{\sigma_1^2 - \sigma_2}{\sigma_1}$$

$$\left(\begin{array}{l} \dfrac{s_2}{s_3}\,\mathbf{Z}^2\overline{\mathcal{Z}} - s_1\,\mathbf{Z}\overline{\mathcal{Z}}^2 + \left(\dfrac{2s_1}{s_3} - \dfrac{s_2^2}{s_3^2}\right)\mathbf{T}\mathbf{Z}^2 + \left(s_1^2 - 2\,s_2\right)\mathbf{T}\overline{\mathcal{Z}}^2 + \\ \qquad \left(\dfrac{s_2 - 3\,s_1^2}{s_3} + \dfrac{s_2^2 s_1}{s_3^2}\right)\mathbf{T}^2\mathbf{Z} + \left(\dfrac{3\,s_2^2 - s_1^2 s_2}{s_3} - s_1\right)\mathbf{T}^2\overline{\mathcal{Z}} + \left(\dfrac{s_1^3}{s_3} - \dfrac{s_2^3}{s_3^2}\right)\mathbf{T}^3 \end{array}\right)$$

Both umbilics belong to the curve. The corresponding asymptotes intersect at X(3448). The real asymptote :

$$\left[\sigma_1\sigma_2^2,\ 2\,\sigma_3\sigma_1^3 - 2\,\sigma_2^3, -\sigma_1^2\sigma_2\sigma_3\right]$$

is parallel to the Euler line. The sixth intersection with the circumcircle is X(1141).

Proof. Direct inspection. □

D sur K001, F=isogD sur K001

$$\begin{array}{rcl} Nd & = & antig(D) = (isg \circ inv \circ isg)\,(D) \\ Nf & = & antig\,(F) = (isg \circ inv)\,(D) \end{array}$$

15.7 Eigentransform

Definition 15.7.1. The mapping $U \mapsto cevadiv\,(U, U_P^*)$ is called eigentransform of U wrt pole P. In ETC, $P = X_6$ is assumed, and notation $ET\,(U)$ is used. Here, the same notation is used, but $P = X_6$ isn't assumed.

Example 15.7.2. Assuming $P = X_6$, pairs (I,J) such that X(J) = ET(X(I)) include :

1	1	13	62	81	3293	174	266	664	2082	1156	1
2	3	14	61	86	3294	190	1	673	1	1492	1
3	1075	19	2128	88	1	512	2142	694	384	1821	1
4	155	20	2130	92	47	648	185	771	1	1942	1941
5	2120	30	2132	94	49	651	1	799	1		
6	194	37	2134	99	39	653	1	811	2083		
7	218	57	2136	100	1	655	1	823	1		
8	2122	63	1712	101	2140	658	1	897	1		
9	2124	69	2138	110	5	660	1	1113	3		
10	2126	75	2172	162	1	662	1	1114	3		

Proposition 15.7.3. *For any point U not on a sideline of triangle ABC, the following properties of eigentransform are easy to verify :*

1. *The barycentrics of $ET\,(U)$ are (cyclically) :*
$$vwp\left(u^2w^2q + u^2v^2r - v^2w^2p\right)$$
2. *$ET(U)$ is the eigencenter of the cevian triangle of U as well as the eigencenter of the anticevian triangle of U_P^*.*
3. *$ET(U) = \sqrt{P} \doteq \sqrt{p} : \sqrt{q} : \sqrt{r}$ (fixed point of the isoconjugacy) if and only if $U = \sqrt{P}$ or U lies on the $CC\left(\sqrt{P}\right)$ circumellipse. When $P = X_6$, then $\sqrt{P} = X_1$ and this locus is the Steiner circumellipse: $yz + zx + xy = 0$.*
4. *Points U, $ET(U)$ and $(ET\,(U))_P^*$ are collinear points of the cubic $p\mathcal{K}\,(P, U)$.*
5. *Points $\sqrt{P}$, $ET(U)$ and $\left(cevadiv\left(U, \sqrt{P}\right)\right)_P^*$ are collinear. The last point is also on the cubic.*

For discussions and generalizations using barycentric coordinates, see Section 1.4 of Ehrmann and Gibert (2009). The cubic Z(U) is there denoted by pK(X6,U), and ET(U) is = (cevian quotient of U and U^*) = (U-Ceva conjugate of U^ -1); ET(U) is the tangential of U^* in pK(X6,U).

15.8 Non pivotal isocubics nK(P,U,k) and nK0(P,U)

Definition 15.8.1. The non pivotal isocubic with pole P, root U and parameter k is defined by the equation :

$$n\mathcal{K}\,(P,U,k) \doteq ux\left(y^2r + qz^2\right) + vy\left(z^2p + rx^2\right) + wz\left(qx^2 + py^2\right) + k\,xyz \tag{15.10}$$

When $k = 0$, the cubic is noted $n\mathcal{K}0\,(P, U)$.

Proposition 15.8.2. *The "third intersections" a $n\mathcal{K}\,(P, U, k)$ with the sidelines are the cocevians of the root U. Therefore, they are aligned. This is to be compared with the fact that, for a $p\mathcal{K}$ cubic, these points are the cevians of the pivot.*

Proof. Direct inspection. □

Remark 15.8.3. In the general case, a $n\mathcal{K}0\,(P, U)$ does not contain F, P, U.

Proposition 15.8.4. *The $NK(X)$ point is the pole of the line $X\,X_P^*$ with respect to the circumconic that passes through X and X_P^* (Bernard Gibert, 2003/10/1). Using barycentrics and $P = p : q : r = f^2 : g^2 : h^2$, we have :*

Proof. Direct computation (see 18.3 for the definition of $crosdiff$). □

Example 15.8.5. Using F=X(1), i.e. P=X(6), we have NK(X(I)) = X(J) for these (I,J):

I	1	2	3	4	6	9	19	31	57	63
J	1	39	185	185	39	2082	2083	2085	2082	2083

Proposition 15.8.6. *When $NK_P(X)$ belongs to the tripolar line of U_P^*, then X belongs to the cubic $n\mathcal{K}0(P,U)$.*

Proof. Direct computation. In Kimberling (1998, p. 240), notation $Z+(XY)$ is used to denote the $n\mathcal{K}0\,(\#1,\,U)$ cubic where the pole U is the isogonal of the tripole of line XY. Therefore,

$$\begin{aligned} Z^+(X_1X_6) &= n\mathcal{K}0\,(\#1,\,513)\\ Z^+(X_3X_6) &= n\mathcal{K}0\,(\#1,\,523)\\ Z^+(X_1X_2) &= n\mathcal{K}0\,(\#1,\,649)\\ Z^+(X_1X_3) &= n\mathcal{K}0\,(\#1,\,650) \end{aligned}$$

□

15.9 Conicopivotal isocubics cK(#F,U)

Definition 15.9.1. A conico-pivotal isocubic $c\mathcal{K}\,(\#F,U)$ (Ehrmann and Gibert, 2009) is a non pivotal isocubic $n\mathcal{K}\,(P,U,k)$ that contains one of the fixed points of the isoconjugacy ($F\neq U$ is assumed). Using $F=f:g:h$ instead of $P=p:q:r=f^2:g^2:h^2$, $k=-2\,(ghu+fhv+fgw)$ and equation becomes :

$$x(gz-hy)^2u+y(fz-hx)^2v+z(fy-gx)^2w=0 \tag{15.11}$$

Proposition 15.9.2. *The pivotal conic is defined as the conic $\mathcal{C}$ tangent to the six lines F_BF_C, AA'_U and cyclically where $F_AF_BF_C$ is the anticevian of F and $A'_UB'_UC'_U$ the cocevian of U. Then the dual conic of $\mathcal{C}$ is conicev $(1/F,1/U)$ and $\mathcal{C}$ itself has equation :*

$$\sum_{cyclic}(gw-hv)^2\,x^2-2\,\left(gu^2h+3\,f\,(gw+vh)\,u+f^2vw\right)zy=0$$

Proof. We have the equations :

$$\begin{aligned} (F_BF_C) &= F_B\wedge F_C=\begin{pmatrix} f\\ -g\\ h\end{pmatrix}\wedge\begin{pmatrix} f\\ g\\ -h\end{pmatrix}=[0,\,2fh,\,2fg]=\left[0,\,\frac{1}{g},\,\frac{1}{h}\right]\\ AA'_U &= U_BU_C=\left[0,\,\frac{1}{v},\,\frac{1}{w}\right] \end{aligned}$$

The equation of $\mathcal{C}$ follows by duality. Barycentrics of the center are $2\,fu-(v+w)\,f-(g+h)\,u$, etc. □

Proposition 15.9.3. *The contact conic (K) is defined as the circumconic whose perspector is*

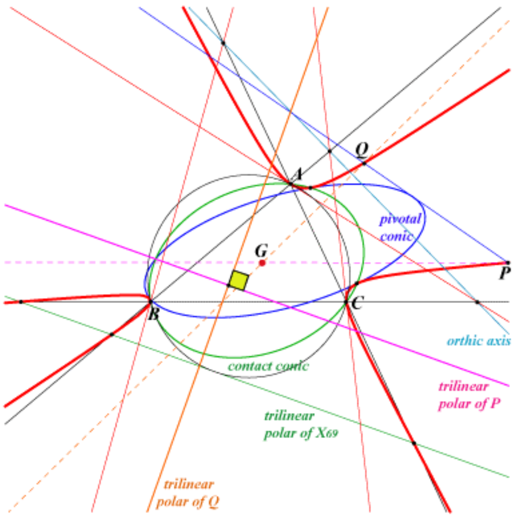
$$K\simeq\left(2\,\frac{f}{u}+\frac{g}{v}+\frac{h}{w}\right)f,\text{ etc}$$

Assuming $F\neq U$, three of the intersections of the pivotal and contact conics are the three contacts of $c\mathcal{K}$ with $\mathcal{C}$, the fourth point being :

$$T_4\simeq\left(2\,\frac{f}{u}+\frac{g}{v}+\frac{h}{w}\right)\div(gw-vh)\,,\text{ etc}$$

Proof. Eliminate z between (K) and $\mathcal{C}$. Obtain $P_1\,(x,y)\;P_3\,(x,y)$, where degrees are respectively 1 and 3. Solving for P_1 gives directly T_4. Eliminate z between $c\mathcal{K}$ and $\mathcal{C}$. This leads again to P_3, proving that each common point is a contact and belongs also to (K). □

Example 15.9.4. A special case is obtained when $U=F$, i.e. when the root is a fixed point of the isoconjugacy. Then $\mathcal{C}=(K)$ is a circumconic. The Tucker cubic K015 is obtained with $F=$X(2), while K228 is obtained with $F=$X(1) and K229 is obtained with $F=$X(6).



Centroid $G = X_2$ is isolated, but belongs nevertheless to the cubic.

Figure 15.3: The Simpson cubic (as depicted in Gibert-CTP)

15.10 Simson cubic, aka K010

Notation 15.10.1. In this section, the involved pole is the centroid, so that $X^* = isot\,(X)$. Due to the nature of the cubic, the key point is $F = X_2$ (involved as fixed point) rather than $P = X_2$ (involved as pole). Therefore, letter P has been used not to describe the pole, but the independent moving point of various parameterizations.

Definition 15.10.2. The **Simson cubic** is the locus of the tripoles of the Simson lines. Depicted as K010 (cf. Figure 15.3) in Gibert (2004-2010). Founding paper is Ehrmann and Gibert (2001).

Proposition 15.10.3. *The Simson cubic (K010) is $c\mathcal{K}\,(\#X_2,\, X_{69})$. Centroid $G = X_2$ belongs to K010 (Simson line of Q is $\mathcal{L}_\infty$ when $Q \in \mathcal{L}_\infty$). Apart from this isolated point, a parameterization of K010 is given in (7.1), where $p : q : r$ are the coefficients of any line orthogonal to the involved Simson line. An other parameterization, using the barycentrics of the involved point on Γ is as follows :*

$$\left(\begin{array}{c} u \\ v \\ w \end{array}\right) \in \Gamma \mapsto \left(\begin{array}{c} b^2wu^2 - c^2vu^2 + \left(b^2 - c^2\right) wvu \\ c^2uv^2 - a^2v^2w + \left(c^2 - a^2\right) wuv \\ a^2vw^2 - w^2b^2u + \left(a^2 - b^2\right) wuv \end{array}\right) \in K010$$

Definition 15.10.4. Cubic K162 is the isogonal transform of the Simpson cubic (that can also be obtained by $Q \mapsto Q \div_b X_6$). Therefore, K162 is $c\mathcal{K}\,(\#X_6,\, X_3)$.

Definition 15.10.5. The **Gibert-Simson transform** is another parameterization of the Simson cubic that also uses an $U \in \Gamma$:

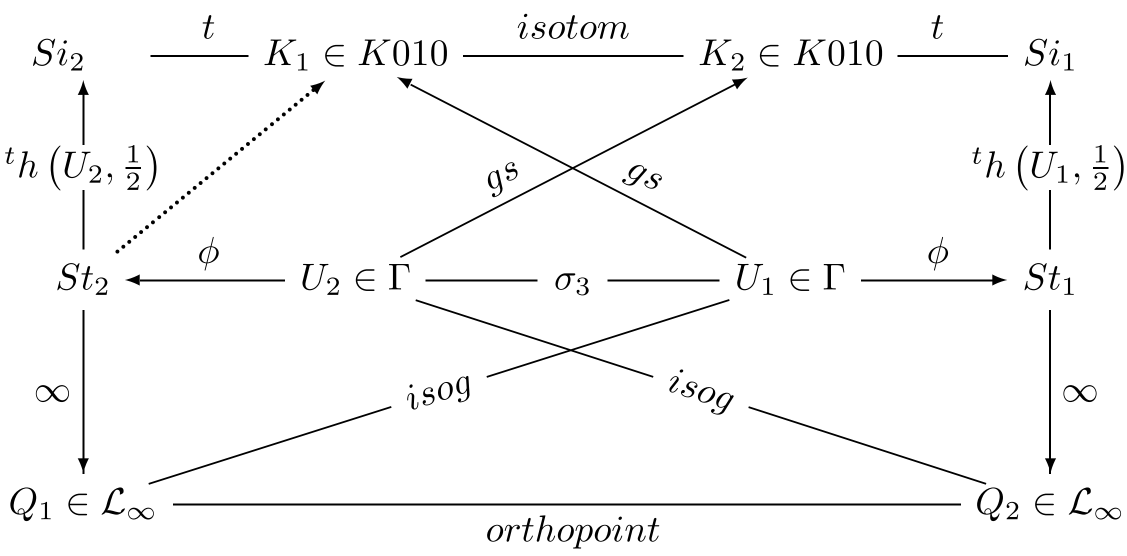
$$GS\,(U) = \text{cyclic}\left[\left(\frac{b^2\left(c^2 + a^2 - b^2\right)}{va^2} - \frac{c^2\left(a^2 + b^2 - c^2\right)}{a^2w}\right) u\right]$$

The lack of uniqueness is due to the binding relation $\sum a^2vw = 0$. Definition introduced in ETC on 2003/10/19, leading to points X(2394) - X(2419) on the Simson cubic and points X(2420)-X(2445) –their isoconjugates– on K162.

Remark 15.10.6. Regarding triangle centers that do not lie on the circumcircle, GS(X(I)) = X(J) for these (I,J): (32,669), (48,1459), (187,1649), (248,879), (485,850), (486,850). Of course, other realizations of $U \mapsto K$ give other results. Here again, only parameterization (7.1) ensures uniqueness.

Example 15.10.7. Use ${}^tP = {}^tX_{525}$ as entry point Figure 15.4 (arrow at the left of the bottom diagram). Obtain $X_{30} = Q_1 \in \mathcal{L}_\infty$ by (5.14), then $X_{74} = U_1 \in \Gamma$ by isogonal conjugacy. The

Steiner line St_1 hasn't received any name, while the Simson line Si_1 of U_1 is ${}^tX_{247}$. The trilinear pole of this line, i.e. $X_{2394} = K_1 \in K010$, can be obtained by $isot \circ ({}^t)$ from Si_1, by gs from U_1 and also directly from P (the dotted line) using parameterization (7.1).



$$where \quad \phi(U) = {}^t isog\left(U_1 *_b X_4\right) \quad and \quad \infty(L) = L \wedge \mathcal{L}_\infty$$

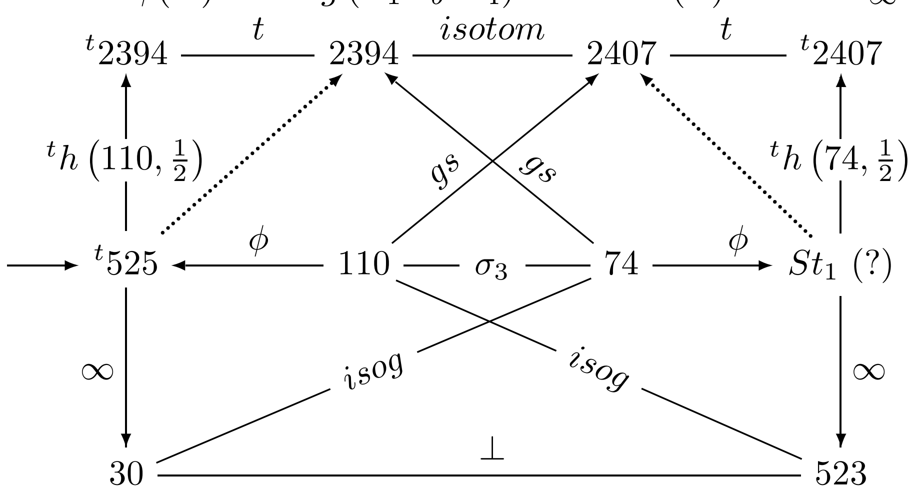


Figure 15.4: The Simson diagram

Proposition 15.10.8 (Fools' Day Theorem). *Direct arrows $U_1 \mapsto K_2$ and $U_2 \mapsto K_1$ have a geometrical meaning : $(gs\,(U))^*$ is the eigencenter of the pedal triangle of U.*

Proof. Point K_2 has no other choice : he *is* the unary cofactor of the pedal triangle of U, and therefore the perspector of this triangle with anything else. □

Proposition 15.10.9. *The barycentric equation of the Simpson cubic is*

$$\sum_{cyclic} x(z^2+y^2)(b^2+c^2-a^2) - 2(a^2+b^2+c^2)x\,y\,z \tag{15.12}$$

In other words, the Simpson cubic is $n\mathcal{K}\left(X_2,\, X_{69},\, -2\left(a^2+b^2+c^2\right)\right)$. Moreover, one of the fixed points (namely $G = X_2$) belongs to the cubic and the Simpson cubic is in fact $c\mathcal{K}\left(\#X_2,\, X_{69}\right)$.

Proof. Obtained from the parametric representation. The converse property is more easily obtained from next coming proposition. □

Definition 15.10.10. Special points wrt K010. Points on sidelines of triangle ABC or of antimedial triangle, together with the centroid are said to be special wrt K010 (because quite every "general" formula turns wrong when dealing with these points).

Proposition 15.10.11. *Among the special points, the following are the sole and only elements of K010 :*
(i) then centroid itself (fixed point under isoconjugacy)
(ii) the vertices of triangle ABC, the cocevians of X_{69} (the root) and points $b+c : b-c : c-b$ or $b-c : b+c : -b-c$ and cyclically.

Proof. Direct inspection. □

Proposition 15.10.12. *When point X is on the Simson cubic but not G, A, B, C then :*
(i) the trilinear polars of X and X^* are perpendicular
(ii) the trilinear polars of X and X^* are concurrent on the nine-point circle
Conversely, when X is not special and either property holds, then X is on the cubic.

Proof. When X is on the Simpson cubic, $tripolar\,(X)$ is a Simson line and conclusion follows from $tripolar = {}^t\circ isot$. Conversely, if (i) then points ${}^tX^* \wedge \mathcal{L}_\infty$ and ${}^tX \wedge \mathcal{L}_\infty$ are the infinity points of both tripolars. They have to be the orthopoint of each other, and (15.12) is re-obtained by elimination. If (ii) then dividing $nineq\,(X \wedge X^*)$ by (15.12), leads to $\prod (y+z)\,/x^2$. When X is on a sideline of ABC, conjugacy is no more defined, and when $y+z=0$ (implying X on the sidelines of the antimedial triangle) then $X \wedge X^*$ is ever $0:1:1$. Outside of these six lines, both conditions are equivalent. □

15.11 Brocard second cubic aka K018

Definition 15.11.1. The Brocard second cubic is inventoried as K018 in (Gibert, 2004-2010). This cubic is $n\mathcal{K}0\,(X_6,\, X_{523})$. It is a circular isogonal focal nK cubic with root X(523) and singular focus X(111). The real asymptote is parallel to GK. It is also the orthopivotal cubic O(X6) and Z+(L) with L = X(3)X(6) in TCCT p.241. See also Z+(O) = CL025 and CL034.

Proposition 15.11.2. *The barycentric equation of K018 is :*

$$\sum_{cyclic} x(b^2 z^2 + y^2 c^2)(b^2 - c^2) = 0 \tag{15.13}$$

Chapter 16

Sondat theorems

16.1 Perspective and directly similar

Remark 16.1.1. When triangle ABC is translated into $A'B'C'$, both triangles are in perspective and the perspector is the direction of the translation. The converse situation is not clear, so that translations will be excluded from what follows, implying the existence of a center. When using composition, we have so examine if nevertheless are reappearing.

Lemma 16.1.2. *When σ is a direct similitude (but not a translation) with center $S = z : 0 : \zeta$ and ratio $k\kappa$ (k is real while κ is unimodular, and $k\kappa \neq \pm 1$) then its matrix in the Morley space is :*

$$\boxed{\sigma} = \begin{pmatrix} k\kappa & \dfrac{z}{t}\,(1 - k\kappa) & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{\zeta}{t}\left(1 - \dfrac{k}{\kappa}\right) & \dfrac{k}{\kappa} \end{pmatrix}$$

One can check that umbilics are fixed points of this transform.

Proof. The characteristic polynomial of matrix :

$$\begin{pmatrix} k\kappa & z_2/t_2 & 0 \\ 0 & 1 & 0 \\ 0 & \zeta_2/t_2 & k/\kappa \end{pmatrix}$$

is $(\mu - 1)\,(\mu - k\,\kappa)\,(\mu + k/\kappa)$. Excluding $k\kappa = \pm 1$ ensures the existence of a center. □

Proposition 16.1.3. *When a direct central similitude $\sigma\,(S,\, k\kappa)$ and a perspector $P \neq S$ are given, the locus $\mathcal{C}$ of points M such that P, M, $M' = \sigma\,(M)$ are collinear is the circle through S, P, $\sigma^{-1}P$.*

Proof. The locus contains certainly the five points such that $M = P$ or $M' = P$ or $M = M'$ i.e. S and both umbilics. The general case results from the fact that $\det\,[P,\, M,\, M']$ is a second degree polynomial in $\mathbf{Z}$, $\overline{\mathcal{Z}}$, $\mathbf{T}$ so that $\mathcal{C}$ is a conic. □

Proposition 16.1.4. *Suppose that triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ are together in perspective (center P) and strictly similar (center S, ratio $k\kappa$, $\kappa \neq \pm 1$). Then P and S are the two intersections of their circumcircles ($P = S$ cannot occur).*

Proof. Use Lubin coordinates relative to $\mathcal{T}_1 = ABC$, and note $S \simeq z : t : \zeta$. Then $A'B'C'$ is obtained as :

$$\boxed{A'B'C'} \simeq \boxed{\sigma} \cdot \boxed{ABC}$$

The determinant of lines AA', BB', CC' factors as :

$$\frac{V dm}{t^2 \kappa^2 \sigma_3}\, k^2\left(\kappa^2 - 1\right)(\kappa - k)\left(\kappa - \frac{1}{k}\right) \times \left(z\,\zeta - t^2\right)$$

proving $S \in \Gamma$. Then perspector is also on this circle (from preceding proposition). We even have the more precise result :

$$P = \left(\begin{array}{ccc} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^{-1} \end{array}\right) \cdot S \quad \text{where} \quad \omega = \frac{1 - k\kappa}{1 - \dfrac{k}{\kappa}}$$

□

Proposition 16.1.5. *Consider a fixed triangle ABC, and describe the plane using the Lubin frame. Consider points $P = \Phi : 1 : 1/\Phi$ and $S = \Theta : 1 : 1/\Theta$ on the unit circle (P as Phi, and S as Sigma. But Sigma is sum, use the next Greek letter). Assume $P \neq S$. Then all triangles $A'B'C'$ that are P-perspective and S-similar to triangle ABC is obtained as follows. Let point O' on the perpendicular bisector of (P, S) be defined by property $(SO, SO') = \kappa$ where κ^2 is a given turn. Draw circle γ centered at O' and going through P and S. Then $A' = \gamma \cap SA$,* etc.

Proof. Point O' can be written as $P + S + x\,(\Theta\Phi : 0 : 1)$. This point is the $\sigma\,(S, k\kappa)$ image of O if and only if :

$$k = \kappa\,\frac{\Phi - \Theta}{\Phi - \kappa^2\Theta},\ \boxed{\sigma} = \left(\begin{array}{ccc} \kappa^2\,(\Phi - \Theta) & \left(1 - \kappa^2\right)\Theta\,\Phi & 0 \\ 0 & \Phi - \kappa^2\Theta & 0 \\ 0 & 1 - \kappa^2 & \Phi - \Theta \end{array}\right)$$

As it should be, σ depends only of κ^2, while the sign of k depends on the choice of κ among the square roots of κ^2. The matrix of circle γ is :

$$\boxed{\gamma} = {}^t\boxed{\sigma}^{-1} \cdot \boxed{\Gamma} \cdot \boxed{\sigma}^{-1} \simeq \left(\begin{array}{ccc} 0 & 1 - \kappa^2 & \kappa^2\Theta - \Phi \\ 1 - \kappa^2 & 2\left(\kappa^2\Phi - \Theta\right) & \left(1 - \kappa^2\right)\Theta\,\Phi \\ \kappa^2\Theta - \Phi & \left(1 - \kappa^2\right)\Theta\,\Phi & 0 \end{array}\right)$$

And it can be checked that P, A, $A' = \sigma\,(A)$ are collinear. □

Proposition 16.1.6. *With same hypotheses, the perspectrix of triangles ABC and $A'B'C'$ is the line :*

$$XYZ = \left[\left(\Phi - \kappa^2\,\Theta\right)\Phi,\ -\Phi\Theta\,\left(\Phi - \kappa^2\,\sigma_1\right) + \kappa^2\,\left(\kappa^2\,\sigma_3 - \sigma_2\Phi\right),\ \kappa^2\,\left(\Phi - \kappa^2\,\Theta\right)\sigma_3\right]$$

Points S, C, C', X, Y are cocyclic (and circularly). When κ^2 reaches Φ/α, then A' moves to P and XYZ becomes the sideline BC. When S is not a vertex A, B, C, the envelope of line XYZ is the Steiner parabola of point S (focus at S, directrix the Steiner line of S). The tangential equation of this parabola is given by matrix :

$$\boxed{\mathcal{P}^*} \simeq \left(\begin{array}{ccc} 2\,\Theta\,\sigma_3 & \sigma_3 & \sigma_2 - \sigma_1\Theta \\ \sigma_3 & 0 & -\Theta \\ \sigma_2 - \sigma_1\Theta & -\Theta & -2 \end{array}\right)$$

Proof. Since XYZ is given by second degree polynomials, the envelope is a conic. It can be obtained by diff and wedge, then eliminate. Parabola comes from the central 0. Focus is obtained in the usual way, and one recognizes S. Directrix Δ is the locus of the reflections of the focus in the tangents. From the special cases, this is the Steiner line of S.

Last point, compute circle (S, X, Y). Special cases $\Theta = \alpha$, etc, and $\Phi = \kappa^2\Theta$ (O' at infinity) are appearing in factor. Otherwise, the equation is :

$$\alpha\,\left(\Phi - \kappa^2\,\Theta\right)\mathbf{Z}\overline{\mathcal{Z}} + \left(\kappa^2\,\alpha - \Phi\right)\left(\mathbf{Z} + \alpha\,\Theta\,\overline{\mathcal{Z}}\right)\mathbf{T} + \left(\Phi\,\Theta - \alpha^2\kappa^2\right)\mathbf{T}^2$$

and this circle goes through A and A'. □

Proposition 16.1.7. *When A, B, C, S, P are according the former hypotheses, let H, H' be the respective orthocenters of ABC and $A'B'C'$. Then midpoint of H, H' belongs to XYZ if and only if $\kappa^2 = -1$ (so that κ is a quarter turn) or :*

$$\kappa^2 = -\Phi^2\,\frac{(\Theta - \sigma_1)}{\sigma_2\,\Theta - \sigma_3},\quad i.e.\,\kappa = (BC,\, OP) + (SA,\, SH)$$

In the second case, XYZ is the perpendicular bisector of (H, H').

Proof. We have $H' = \sigma(H)$ and equation in κ^2 is straightforward. Then we have :

$$\omega^2(BC) = -\beta\gamma,\, \omega^2(OP) = \Phi^2,\, \omega^2(SA) = -\Theta\alpha,\, \omega^2(SH) = -\sigma_3\Theta\,\frac{\Theta-\sigma_1}{\sigma_2\Theta-\sigma_3}$$ □

Corollary 16.1.8. *Start from triangle ABC, and assume that* $XYZ = [\rho, \sigma, \tau]$ *while* κ *is a quarter turn. Then* X, Y, Z *are* $X = 0 : \tau : -\sigma$, etc. *Lines* $X\delta_A = [S_c\sigma + S_b\tau, a^2\sigma, a^2\tau]$, etc *are the perpendicular at* X *to* BC, etc. *Finally, point* A' *is* $Y\delta_B \cap Z\delta_C$, etc. *In other words :*

$$A' \simeq \begin{pmatrix} \rho\left(\rho S_b S_c + \sigma S_c S_a + \tau S_a S_b\right) - 4\,S^2\sigma\tau \\ b^2\rho\left(\tau c^2 - \sigma S_a - \rho S_b\right) \\ c^2\rho\left(\sigma b^2 - \tau S_a - \rho S_c\right) \end{pmatrix}, \text{ etc}$$

The perspector and the similicenter are :

$$P = isogon\left(\boxed{\mathcal{M}}\cdot{}^t\Delta\right)\ ;\ S = isogon\left((\sigma-\tau)\,\rho : \sigma\,(\tau-\rho) : \tau\,(\rho-\sigma)\right)$$

while the ratio of the similitude is :

$$k = \frac{\left(a^2+b^2+c^2\right)\rho\sigma\tau - S_a\rho\left(\sigma^2+\tau^2\right) - S_b\sigma\left(\rho^2+\tau^2\right) - S_c\tau\left(\rho^2+\sigma^2\right)}{2\,S\,(\sigma-\tau)\,(\rho-\tau)\,(\rho-\sigma)}$$

Proof. Straightforward computation. □

16.2 Perspective and inversely similar

Lemma 16.2.1. *A circumscribed rectangular hyperbola goes through* A, B, C, H, Gu *where* H *is the orthocenter and* Gu *is the gudulic point, the intersection of the RH and the circumcircle. Directions of axes are given by the bisectors of, for example,* AGu *and* BC. *Then directions of asymptotes are obtained by a* 45° *rotation (or taking again the bisectors).*

Lemma 16.2.2. *When a rectangular hyperbola* $\mathcal{H}$ *is known by its implicit equation*

$$\kappa^2\overline{\mathcal{Z}}^2 - \frac{1}{\kappa^2}\,\mathbf{Z}^2 + \left(W\,\overline{\mathcal{Z}} + V\,\mathbf{Z}\right)\mathbf{T} + Q\,\mathbf{T}^2$$

then points $M \in \mathcal{H}$ *can be parametrized as :*

$$M = \frac{1}{2}\begin{pmatrix} +V\,\kappa^2 \\ 1 \\ -W\,\dfrac{1}{\kappa^2} \end{pmatrix} + X\begin{pmatrix} \kappa \\ 0 \\ \dfrac{1}{\kappa} \end{pmatrix} + Y\begin{pmatrix} +i\,\kappa \\ 0 \\ -i\,\dfrac{1}{\kappa} \end{pmatrix} \tag{16.1}$$

where X, Y *are real quantities linked by :*

$$YX = \frac{-i}{16}\left(\kappa^2V^2 + 4\,Q - \frac{W^2}{\kappa^2}\right)$$

Proof. This way of writing may look weird but, most of the time, hyperbola equations are appearing that way. □

Lemma 16.2.3. *Consider four points* M_j *on a rectangular hyperbola, parametrized by* (16.1). *These points form an orthocentric quadrangle if and only if :*

$$256\,x_1x_2x_3x_4 = \left(\kappa^2V^2 + 4\,Q - \frac{W^2}{\kappa^2}\right)^2$$

Proof. We write that $(M_1 \wedge M_2)\cdot\boxed{\mathcal{M}_z}\cdot{}^t(M_3 \wedge M_4) = 0$, and obtain this condition. The conclusion follows from the symmetry of the the result. □

Lemma 16.2.4. *When ψ is a central inverse similitude (reflections in a line are allowed, but not the other isometries), then its matrix in the Morley space can be written as :*

$$\boxed{\psi} = \left(\begin{array}{ccc} 0 & \dfrac{z}{t} - k\,\kappa^2\,\dfrac{\zeta}{t} & k\kappa^2 \\ 0 & 1 & 0 \\ \dfrac{k}{\kappa^2} & \dfrac{\zeta}{t} - \dfrac{k}{\kappa^2}\,\dfrac{z}{t} & 0 \end{array}\right)$$

Point $S = z : 0 : \zeta$ is the center, axes are directed by $\pm\kappa^2$ and ratio is k (κ is unimodular while k is real and $k = \pm 1$ is a dubious case). One can check that umbilics are exchanged by this transform.

Proof. Let $z_2 : t_2 : \zeta_2$ be the image of the origin $0 : 1 : 0$. The characteristic polynomial of matrix :

$$\left(\begin{array}{ccc} 0 & z_2/t_2 & k\,\kappa^2 \\ 0 & 1 & 0 \\ \dfrac{k}{\kappa^2} & \zeta_2/t_2 & 0 \end{array}\right)$$

is $(\mu - 1)\,(\mu - k)\,(\mu + k)$. Excluding $k = \pm 1$ ensures the existence of a center. Consider the reflection δ about line through S and $\kappa^2 : 0 : 1$. We have :

$$\boxed{\delta} = subs\left(k = \pm 1, \boxed{\psi}\right) \quad ; \quad \boxed{\psi} \cdot \boxed{\delta} = \boxed{\delta} \cdot \boxed{\psi} = \left(\begin{array}{ccc} k & (1-k)\,\dfrac{z}{t} & 0 \\ 0 & & 0 \\ 0 & (1-k)\,\dfrac{\zeta}{t} & k \end{array}\right)$$

□

Remark 16.2.5. The unimodular κ was an intrinsic quantity when we were dealing with direct similitudes. Now, κ^2 measure the angle between the real axis and one of the axes of the antisimilitude.

Proposition 16.2.6. *When a central antisimilitude $\psi\,(S, k\kappa)$ and a perspector $P \neq S$ are given, the locus $\mathcal{H}$ of points M such that P, M, $M' = \psi\,(M)$ are collinear is the conic through S, P, $\psi^{-1}P$ and directions of the ψ-axes (and this conic is a rectangular hyperbola).*

Proof. The locus contains certainly the five points such that $M = P$ or $M' = P$ or $M = M'$ i.e. S and both directions $\pm\kappa^2 : 0 : 1$. The general case results from the fact that $\det[P, M, M']$ is a second degree polynomial in $\mathbf{Z}$, $\overline{\mathcal{Z}}$, $\mathbf{T}$ so that $\mathcal{C}$ is a conic. □

Proposition 16.2.7. *Suppose that triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ are together in perspective (center P) and strictly antisimilar (center S, ratio $k \neq \pm 1$, direction of axes $\pm\kappa^2 : 0 : 1$). Let X be one of the points S, P, $\pm\kappa^2 : 0 : 1$. Then the other three are obtained as the remaining intersections between conic $\mathcal{H}$ through A, B, C, H, X and conic $\mathcal{H}'$ through A', B', C', H', X.*

Proof. Use Lubin coordinates relative to $\mathcal{T}_1 = ABC$, and note $S \simeq z : t : \zeta$. Then $A'B'C'$ is obtained as :

$$\boxed{A'B'C'} \simeq \boxed{\psi} \cdot \boxed{ABC}$$

The determinant of lines AA', BB', CC' factors as :

$$\frac{V dm}{t^2 \sigma_3}\, k\left(k^2 - 1\right) \times conic$$

proving that S belongs to the rectangular hyperbola whose implicit and parametric equations are :

$$\frac{-1}{\kappa^2}\,\mathbf{Z}^2 + \kappa^2 \overline{\mathcal{Z}}^2 + \left(\frac{\sigma_1}{\kappa^2} - \frac{\kappa^2}{\sigma_3}\right)\mathbf{T}\mathbf{Z} + \left(\frac{\sigma_2 \kappa^2}{\sigma_3} - \frac{\sigma_3}{\kappa^2}\right)\mathbf{T}\overline{\mathcal{Z}} + \left(\frac{\sigma_1 \kappa^2}{\sigma_3} - \frac{\sigma_2}{\kappa^2}\right)\mathbf{T}^2$$

$$M\,(x) \simeq \left[\begin{array}{c} x\,\kappa - \dfrac{\kappa^4 - \sigma_1 \sigma_3}{2\,\sigma_3} + \dfrac{1}{16\,x}\,\kappa\left(\kappa^2\left(\dfrac{\kappa^2}{\sigma_3} + \dfrac{\sigma_1}{\kappa^2}\right)^2 - \dfrac{1}{\kappa^2}\left(\dfrac{\sigma_3}{\kappa^2} + \dfrac{\sigma_2 \kappa^2}{\sigma_3}\right)^2\right) \\ 1 \\ \dfrac{1}{\kappa}x + \dfrac{\sigma_2 \kappa^4 - \sigma_3^2}{2\,\kappa^4 \sigma_3} - \dfrac{1}{16\,x}\,\dfrac{1}{\kappa}\left(\kappa^2\left(\dfrac{\kappa^2}{\sigma_3} + \dfrac{\sigma_1}{\kappa^2}\right)^2 - \dfrac{1}{\kappa^2}\left(\dfrac{\sigma_3}{\kappa^2} + \dfrac{\sigma_2 \kappa^2}{\sigma_3}\right)^2\right) \end{array}\right]$$

Then perspector is also on this hyperbola (from preceding proposition). We even have the more precise result :

$$k = \frac{x_S - x_P}{x_S + x_P}$$

□

Proposition 16.2.8. *Consider a triangle A, B, C, its orthocenter H and two points S, P such that the six points A, B, C, H, P, S are on the same conic $\mathcal{H}$. Then it exists exactly one triangle $A'B'C'$ that is together S-antisimilar and P perspective with ABC. Moreover A', B', C', H', P, S are on the same conic $\mathcal{H}'$, both conics are rectangular hyperbolae and share the same asymptotic directions. Finally, the gudulic point G_u of conic $\mathcal{H}$ sees triangle ABC at right angles with trigon $A'B'C'$, and the fourth intersection of conic $\mathcal{H}'$ with circle Γ' sees triangle $A'B'C'$ at right angles with trigon ABC.*

Proof. A conic through A, B, C, D is a rectangular hyperbola. Consider one of its asymptotes and draw a parallel Δ to this line through point S. Let A'', B'', C'', H'' be the reflections of A, B, C, H into Δ. Then we have A'=SA"∩PA, etc. Final result comes from □

Proposition 16.2.9. *When A, B, C, H, P are fixed, the direction of the perspectrix XYZ is also fixed.*

Proposition 16.2.10. *When A, B, C, H, S are fixed and P moves onto the A, B, C, H, S hyperbola, the envelope of the perspectrix XYZ is the parabola inscribed in triangle ABC whose directrix is line HS.*

16.3 Orthology, general frame

Definition 16.3.1. We say that point P sees triangle $A'B'C'$ at right angles to trigon ABC when P is different from A', B', C' and verifies $PA' \perp BC$ etc.

Remark 16.3.2. Should point P be at infinity, all sidelines of ABC would have the same direction, and triangle ABC would be degenerated (flat). Some problems have to be expected...

Lemma 16.3.3. *The orthodir of the BC sideline is :$\delta_A \doteq a^2 : -S_c : -S_b$. This is also the direction of line HA, where H is the orthocenter of the triangle.*

Proposition 16.3.4. *Consider two non degenerate finite triangles ABC, $A'B'C'$ and suppose that it exists a finite point P, different from A', B', C', that sees triangle $A'B'C'$ at right angles to trigon ABC. Then it exist a point U that sees triangle ABC at right angles to trigon $A'B'C'$.*

Proof. From hypothesis, A' belongs to line $P\delta_A$. Then it exists a real number $k_A \neq \infty$ such that $A' = k_A P + \delta_A$. And the same for B', C'. Now compute :

$$A'' \doteq \boxed{\mathcal{M}}^{t} . \; (B' \wedge C') = -2S\,(p+q+r) \begin{pmatrix} -k_B - k_C \\ k_B \\ k_C \end{pmatrix}$$

This comes from $P \wedge P = 0$, together with $\boxed{\mathcal{M}} . {}^t(\delta_B \wedge \delta_C) = \boxed{\mathcal{M}} . {}^t\mathcal{L}_\infty = 0$. Since $p + q + r \neq 0$ and $k_j \neq 0$ is assumed, column A'' really defines a direction. We can therefore simplify and obtain :

$$A''B''C'' = \begin{pmatrix} -k_B - k_C & k_A & k_A \\ k_B & -k_A - k_C & k_B \\ k_C & k_C & -k_A - k_B \end{pmatrix}$$

This triangle is perspective to ABC, with perspector $U = k_A : k_B : k_C$. Therefore point U sees triangle ABC at right angles to trigon $A'B'C'$. □

Definition 16.3.5. We say that two triangles are orthologic to each other when it exists a point P that sees triangle $A'B'C'$ at right angles to trigon ABC and a point U that sees triangle ABC at right angles to trigon $A'B'C'$. We also say that P, U are the orthology centers of the triangles (P looking at $A'B'C'$, and U looking at ABC). In this definition, flat triangles and centers at infinity are allowed.

Remark 16.3.6. This property cannot be reworded in a shorter form (the so-called symmetry), since $P \notin \mathcal{L}_\infty$is required to be sure of the existence of U, but this is not sufficient to be sure of $U \notin \mathcal{L}_\infty$.

Proposition 16.3.7. *Assume that triangle ABC is not degenerate and let points finite points P, U by described their barycentrics $P = p : q : r$ and $U = u : v : w$ (here $P = U$ is allowed). Moreover, assume that U is not on the sidelines of ABC. Then all triangles $A'B'C'$ such that point P sees triangle $A'B'C'$ at right angles to trigon ABC and point U sees triangle ABC at right angles to trigon $A'B'C'$ are given by formula :*

$$\boxed{A'B'C'} \simeq \left(\begin{array}{ccc} pu + a^2\vartheta & pv - S_c\vartheta & pw - S_b\vartheta \\ qu - S_c\vartheta & qv + b^2\vartheta & qw - S_a\vartheta \\ ru - S_b\vartheta & rv - S_a\vartheta & rw + c^2\vartheta \end{array}\right) = \left(P \cdot {}^tU\right) + 2S\vartheta \boxed{\mathcal{M}} \tag{16.2}$$

where ϑdescribes an homothecy centered at P.

Proof. Since $u \neq 0$, relation $A' \in P\,\delta_A$ can be written as $uP + \vartheta_A\delta_A$, and the same holds for the other points. Now, compute :

$$\det\left[uP + \vartheta_A\delta_A,\, vP + \vartheta_B\delta_B,\, \boxed{\mathcal{M}}^t (C \wedge U)\right] = 2uvS\,(p + q + r)\,(\vartheta_A - \vartheta_B)$$

Due to 0the hypotheses, all the ϑ must have the same value, leading to the formula. Converse is obvious, when a triangle is as described by the formulas, both orthologies are verified. □

Remark 16.3.8. If P were at infinity, PA', PB', PC' would have the same direction, and also the sidelines of ABC. In the formula, this would lead to A', B', C' at infinity. If coordinate u was 0 then $UB \parallel UC$ so that $A'B' \parallel A'C'$. In the formula, this would lead to A'at infinity.

16.4 Simply orthologic and perspective triangles

Definition 16.4.1. When triangles are orthologic with $P \neq U$, we say they are simply orthologic. When $P = U$, we say they are bilogic.

Notation 16.4.2. In this chapter, when triangles ABC and $A'B'C'$ are in perspective, their perspector will be noted Ω (and never P, nor U), while the perspectrix will be noted XYZ with $X = BC \cap B'C'$, etc.

Theorem 16.4.3 (First Sondat Theorem)**.** *Assume that triangle ABC is finite and non degenerate ; P is at finite distance ; U is different from P and is not on the sidelines. Then it exists exactly one triangle $A'B'C'$ such that (1) P sees $A'B'C'$ at right angles to trigon ABC (2) U sees ABC at right angle to trigon $A'B'C'$ (3) $A'B'C'$ is perspective to ABC. When $A'B'C'$ is chosen that way, the corresponding perspector Ω is collinear with points P, U and the perspectrix XYZ is orthogonal to PU.*

Proof. We have assumed that $P = p : q : w$ is different from $U = u : v : w$. Then ϑ is fixed by the condition of being perspective and we have :

$$\begin{aligned}
\vartheta &= -\frac{u\,(vS_b - wS_c)\,qr + v\,(wS_c - uS_a)\,pr + w\,(uS_a - vS_b)\,pq}{(vS_b - wS_c)\,pS_a + (wS_c - uS_a)\,qS_b + (uS_a - vS_b)\,rS_c} \\
\Omega &= \frac{S_cw - S_bv}{qw - rv} : \frac{S_cw - S_au}{pw - ru} : \frac{S_bv - S_au}{pv - qu} \\
tripolar\,(\Delta) &= \left(\begin{array}{c} \dfrac{(S_cw - S_bv)\,p + \left(S_bu + a^2w\right)q - \left(S_cu + a^2v\right)r}{S_bq - S_cr} \\ \dfrac{(S_au - S_cw)\,q + \left(b^2u + S_cv\right)r - \left(S_av + b^2w\right)p}{S_cr - S_ap} \\ \dfrac{(S_bv - S_au)\,r + \left(c^2v + S_aw\right)p - \left(S_bw + c^2u\right)q}{S_ap - S_bq} \end{array}\right)
\end{aligned}$$

First assertion is proved by $\Omega \simeq \alpha\,U - \beta\,P$ where :

$$\left(\begin{array}{c} \alpha \\ \beta \end{array}\right) \simeq \left[\begin{array}{c} (qw - rv)\,puS_a + (ru - pw)\,vqS_b + rw\,(pv - qu)\,rwS_c \\ (qw - rv)\,u^2S_a + (ru - pw)\,v^2S_b + (pv - qu)\,w^2S_c \end{array}\right]$$

Second assertion is proved by checking that

$$\Delta \cdot \boxed{OrtO} \cdot (normalized\,(P) - normalized\,(U)) = 0$$

□

16.5 Bilogic triangles

Proposition 16.5.1. *Bilogic triangles are ever in perspective. When orthology center is $U = u : v : w$ is finite and not on the sidelines, formula (16.2) becomes*

$$\boxed{A'B'C'} \simeq \left(\begin{array}{ccc} u^2 + a^2\vartheta & uv - S_c\vartheta & uw - S_b\vartheta \\ uv - S_c\vartheta & v^2 + b^2\vartheta & vw - S_a\vartheta \\ uw - S_b\vartheta & vw - S_a\vartheta & r^2 + c^2\vartheta \end{array} \right) = \left(U \cdot {}^tU\right) + 2S\vartheta \boxed{\mathcal{M}} \tag{16.3}$$

while the perspector and the tripole of the perspectrix are respectively :

$$\Omega \simeq (\vartheta\, vw - S_a)^{-1} : (\vartheta\, wu - S_b)^{-1} : (\vartheta\, uv - S_c)^{-1}$$

$$tripolar\,(XYZ) \simeq \left(\begin{array}{c} \left(vwa^2 + u\,(vS_b + wS_c - uS_a)\right)\vartheta - 4\,S^2 \\ \left(uwb^2 + v\,(wS_c + uS_a - vS_b)\right)\vartheta - 4\,S^2 \\ \left(uvc^2 + w\,(uS_a + vS_b - wS_c)\right)\vartheta - 4\,S^2 \end{array} \right)$$

Proof. Straightforward computation. □

Proposition 16.5.2. *When triangle ABC is fixed and the bilogic center U is given, the locus of the perspector Ω of the bilogic triangles $A'B'C'$ is the rectangular hyperbola that goes through A, B, C, U, H =X(4). The perspector of this circumconic is :*

$$u\,(vS_b - wS_c) : v\,(wS_c - uS_a) : w\,(uS_a - vS_b)$$

while the envelope of the perspectrix is the inconic whose perspector is :

$$(vS_b - wS_c)^{-1} : (wS_c - uS_a)^{-1}, (uS_a - vS_b)^{-1}$$

Proof. Eliminate ϑ from Ω. Then eliminate ϑ from XYZ and take the adjoint matrix. □

Proposition 16.5.3. *Let ABC, $A'B'C'$ be two bilogic triangles, with orthology center U, perspector Ω and perspectrix (XYZ) where $X = BC \cap B'C'$, etc. Then lines $U\Omega$ and XYZ are orthogonal. Moreover we have $(UX) \perp (AA'\Omega)$, etc.*

Proof. We compute the orthodir δ of line XYZ and find that :

$$\delta = \left[\begin{array}{c} \left(u^2\,(w+v)\,S_a - S_c uw^2 - S_b v^2 u\right)\theta + S_b S_c\,(v+w) - a^2 S_a u \\ \left(v^2\,(w+u)\,S_b - S_a vu^2 - S_c vw^2\right)\theta + S_c S_a\,(w+u) - b^2 S_b v \\ \left(w^2\,(v+u)\,S_c - S_a wu^2 - S_b wv^2\right)\theta + S_a S_b\,(u+v) - c^2 S_c w \end{array} \right]$$

It remains to check that $normalized\,(\Omega) - normalized\,(U)$ is proportional to δ. □

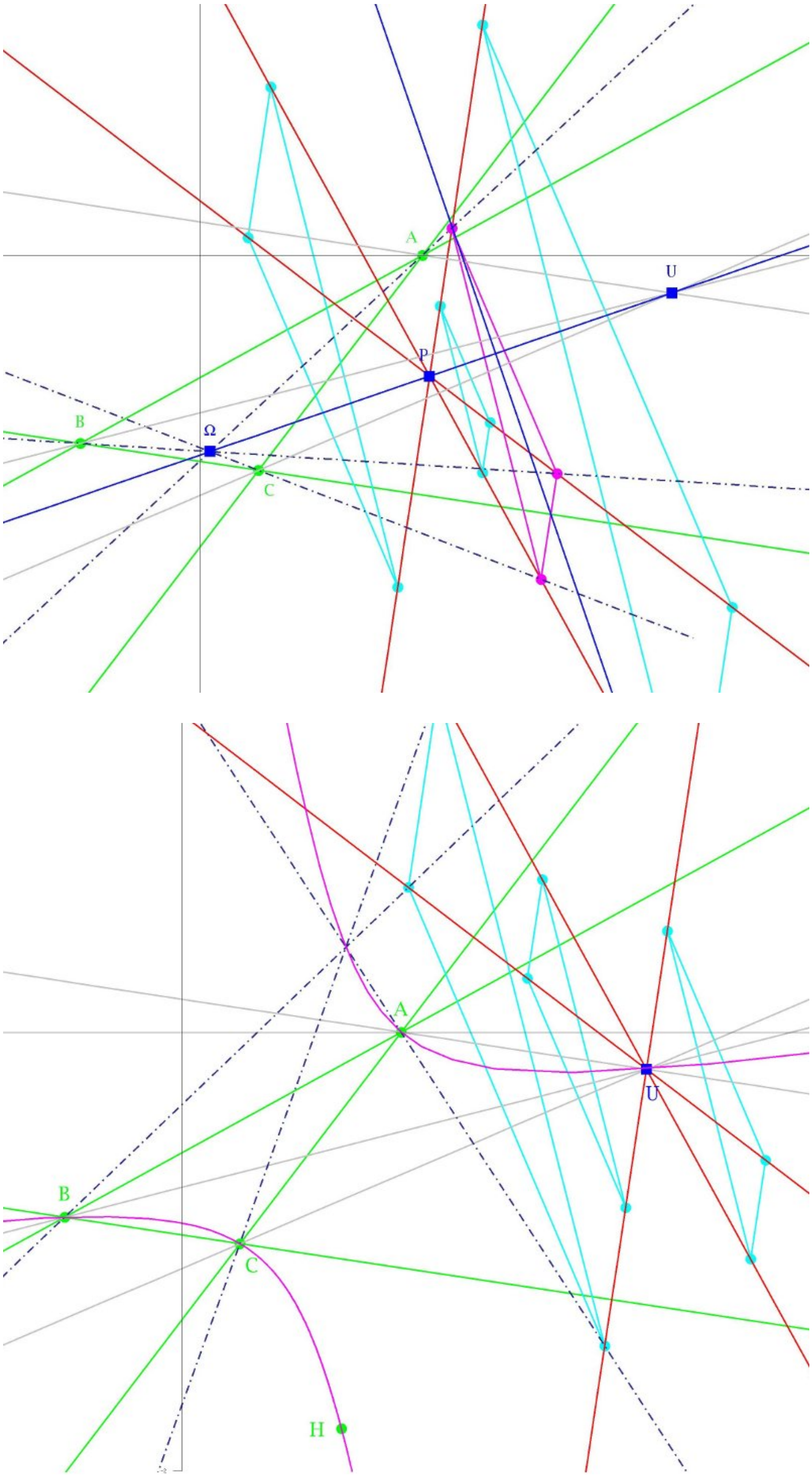


Figure 16.1: Orthology and perspective.

Chapter 17

Special Triangles

Central triangles have been defined in Section 2.2.

17.1 Changing coordinates, functions and equations

Proposition 17.1.1. *Any triangle $\mathcal{T}$ can be used as a barycentric basis instead of triangle ABC. When columns of triangle $\mathcal{T}$ are synchronized, the old barycentrics $x : y : z$ (relative to ABC) can be obtained from the new ones $\xi : \eta : \zeta$ (relative to $\mathcal{T}$) by :*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boxed{\mathcal{T}} \cdot \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

while the converse transformation can be done using the adjoint matrix.

Proof. A column of synchronized barycentrics acts on the matrix of rows containing the projective coordinates of the vertices of the reference triangle by the usual matrix multiplication. □

Remark 17.1.2. By definition of synchronized barycentrics, $\mathcal{L}_\infty \cdot \boxed{\mathcal{T}} \simeq \mathcal{L}_\infty$ and the line at infinity is (globally) invariant.

Proposition 17.1.3. *Let α, β, γ the side lengths of triangle $\mathcal{T}$ (computed using Theorem 5.2.4). Consider a central punctual transformation Φ that can be written as :*

$$p : q : r \mapsto u : v : w = \phi\,(a, b, c, p, q, r)$$

with all the required properties of symmetry and homogeneity. Consider now the corresponding punctual transformation Φ' with respect to triangle $\mathcal{T}$ (written in its normalized form) and define $\phi_\mathcal{T}$ as the action of Φ' on the old barycentrics (the ones related to ABC). Then :

$$\phi_\mathcal{T}\left(a, b, c, \begin{pmatrix} p \\ q \\ r \end{pmatrix}\right) = \mathcal{T} \cdot \phi\left(\alpha, \beta, \gamma, \mathcal{T}^{-1} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}\right)$$

Example 17.1.4. Applied to the isogonal transform and some usual triangle, this leads to formulas given in Figure 17.1. The term "complementary conjugate" is a synonym for "medial isogonal conjugate", as is "anticomplementary conjugate" for "anticomplementary isogonal conjugate". Also, "excentral isogonal conjugate" is "X(188)-aleph conjugate" and "orthic isogonal conjugate" is "X(4)-Ceva conjugate".

Proposition 17.1.5. *Let α, β, γ the side lengths of triangle $\mathcal{T}$. Consider a conic Φ whose matrix can be written as $M\,(a, b, c)$ with the required properties of symmetry and homogeneity. Consider now the corresponding conic Φ' with respect to triangle $\mathcal{T}$ (written in its normalized form) and define $M_\mathcal{T}$ as the matrix defining Φ' wrt the old barycentrics. Then :*

$$M_\mathcal{T}(a, b, c) = {}^t\mathcal{T}^{-1} \cdot M\,(\alpha, \beta, \gamma)\ \cdot \mathcal{T}^{-1}$$

Example 17.1.6. Applied to the circumcircle and some usual triangles, this leads to formulas given in Figure 17.1.

17.2 Some central triangles (to be completed...)

17.2.1 Medial triangle

definition cevian triangle of the centroid X(2)

side_length (strong values)

$$[\alpha, \beta, \gamma] = \frac{1}{2}\,[a,\, b,\, c]$$

barycentrics (normalized)

$$\boxed{\mathcal{C}_2} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

17.2.2 Antimedial triangle

definition anticevian triangle of the centroid X(2)

side_length (strong values)

$$[\alpha, \beta, \gamma] = 2\,[a,\, b,\, c]$$

barycentrics (normalized)

$$\boxed{\mathcal{A}_2} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

17.2.3 Incentral triangle

definition cevian triangle of the incenter X(1)

pythagoras (strong values)

$$\alpha^2 = \frac{abc\,\left(a^3 + a^2 b + a^2 c - ab^2 - ac^2 + 3\,abc - b^3 + b^2 c + bc^2 - c^3\right)}{(a+b)^2\,(a+c)^2}$$

barycentrics (normalized)

$$\boxed{\mathcal{T}} = \begin{pmatrix} 0 & \frac{a}{a+c} & \frac{a}{a+b} \\ \frac{b}{b+c} & 0 & \frac{b}{a+b} \\ \frac{c}{b+c} & \frac{c}{a+c} & 0 \end{pmatrix}$$

17.2.4 Orthic triangle

definition cevian triangle of the orthocenter X(4)

side_length (strong values)

$$[\alpha, \beta, \gamma] = \frac{1}{abc}\,\left[a^2 S_a,\, b^2 S_b,\, c^2 S_c\right]$$

barycentrics (normalized)

$$\boxed{\mathcal{C}_4} = \begin{pmatrix} 0 & S_c/b^2 & S_b/c^2 \\ S_c/a^2 & 0 & S_a/c^2 \\ S_b/a^2 & S_a/b^2 & 0 \end{pmatrix}$$

angles $\pi - 2A,\, \pi - 2B,\, \pi - 2C$

17.2.5 Intouch triangle (contact triangle)

definition Cevian triangle of the Gergonne point X(7).

pythagoras (strong equality)

$$\left[\alpha^2,\beta^2,\gamma^2\right]=\frac{\rho}{2R}\left[a\left(b+c-a\right),\,b\left(c+a-b\right),\,c\left(a+b-c\right)\right]$$

thus similar with the excentral triangle (ratio $\rho/2R$)

barycentrics (normalized)

$$\boxed{\mathcal{C}_7}=\left(\begin{array}{ccc}0 & \dfrac{a+b-c}{b} & \dfrac{c+a-b}{c}\\ \dfrac{a+b-c}{a} & 0 & \dfrac{b-a+c}{c}\\ \dfrac{c+a-b}{a} & \dfrac{b-a+c}{b} & 0\end{array}\right)$$

17.2.6 Residual triangles

Definition 17.2.1. The residuals triangle associated with point P are the triangles AP_bP_c, P_aBP_c, P_aP_bC where $P_aP_bP_c$ is the cevian triangle of P (mind the order...).

Proposition 17.2.2. *The A-residual of the orthic and the intouch triangles have the following sidelengths :*

$$\begin{aligned}\left[\alpha,\,\beta,\,\gamma\right]_{orthic} &= \frac{S_a}{bc}\left[a,\,b,\,c\right]\\ \left[\alpha^2,\,\beta^2,\,\gamma^2\right]_{intouch} &= \frac{\left(b+c-a\right)^2}{4\,bc}\left[\left(a+b-c\right)\left(a-b+c\right),\,bc,\,bc\right]\end{aligned}$$

The incenters of the orthic residuals are the orthocenters of the intouch residuals.

extra triangles the side lengths of the extra triangles are proportional to $a:b:c$

17.2.7 Extouch triangle

definition cevian triangle of the Nagel point X(8).

side_length (strong values)

$$[\alpha,\beta,\gamma]=$$

barycentrics (normalized)

$$\boxed{\mathcal{T}}=$$

17.2.8 Excentral triangle

definition anticevian triangle of the incenter

pythagoras (strong values)

$$\left[\alpha^2,\beta^2,\gamma^2\right]=\frac{2R}{\rho}\left[a\left(b+c-a\right),\,b\left(c+a-b\right),\,c\left(a+b-c\right)\right]$$

thus similar with the intouch triangle (ratio $2R/\rho$)

barycentrics (normalized)

$$\boxed{\mathcal{T}}=\left(\begin{array}{ccc}\dfrac{-a}{b+c-a} & \dfrac{a}{c+a-b} & \dfrac{a}{a+b-c}\\ \dfrac{b}{b+c-a} & \dfrac{-b}{c+a-b} & \dfrac{b}{a+b-c}\\ \dfrac{c}{b+c-a} & \dfrac{c}{c+a-b} & \dfrac{-c}{a+b-c}\end{array}\right)$$

17.2.9 Tangential triangle

definition Anticevian triangle of X(6). The sidelines are the tangents to the ABC-circumcircle at the vertices.

side_length (strong values)

$$[\alpha, \beta, \gamma] = \frac{abc}{2\, S_a S_b S_c}\; \left[a^2 S_a,\, b^2 S_b,\, c^2 S_c\right]$$

Therefore, this triangle is similar to the orthic triangle.

barycentrics (normalized)

$$\boxed{\mathcal{A}_6} = \begin{pmatrix} -\dfrac{a^2}{S_a} & \dfrac{a^2}{S_b} & \dfrac{a^2}{S_c} \\ \dfrac{b^2}{S_a} & -\dfrac{b^2}{S_b} & \dfrac{b^2}{S_c} \\ \dfrac{c^2}{S_a} & \dfrac{c^2}{S_b} & -\dfrac{c^2}{S_c} \end{pmatrix}$$

17.2.10 Fuhrmann triangle

Definition 17.2.3. Fuhrmann triangle is $A''B''C''$ where $A'B'C'$ is the circumcevian triangle of X_1 and A'' is the reflection of A' in sideline BC (and cyclically for B'' and C'').

Proposition 17.2.4. *Side length of Fuhrmann triangle are :*

$$W_4 \times \left(\sqrt{\frac{a}{(a+b-c)\,(a-b+c)}}, \sqrt{\frac{b}{(b-c+a)\,(b+c-a)}}, \sqrt{\frac{c}{(c+a-b)\,(c-a+b)}} \right)$$
$$\text{where} \quad W_4 = \sqrt{a^3+b^3+c^3-(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2)+3\,abc}$$

Quantity W_4 is the Fuhrmann square root (10.14)*. Circumcircle of the Fuhrmann triangle is the Fuhrmann circle of ABC (whose diameter is* $[X_4,\, X_8]$

Remark 17.2.5. As noticed in (Dekov, 2007), $OIFuNa$ and $OIHFu$ are parallelograms, whose respective centroids are the Spieker center and nine-point center, respectively, where $IFu = 2NI$ or $IFu = R - 2r$.

17.2.11 Star triangle

Definition 17.2.6. Star triangle. Consider the midpoints $A'B'C'$ of the sidelines of triangle ABC. Draw from each midpoint the perpendicular line to the corresponding bisector. These three lines determine a triangle $A^*B^*C^*$. This is our star.

Proposition 17.2.7. *The synchronized barycentrics and the sidelengths of the star triangle are :*

$$\boxed{\mathcal{T}_*} \simeq \begin{pmatrix} b+c & c-b & b-c \\ c-a & c+a & a-c \\ b-a & a-b & b+a \end{pmatrix} \cdot \begin{pmatrix} b+c-a & 0 & 0 \\ 0 & c+a-b & 0 \\ 0 & 0 & a+b-c \end{pmatrix}^{-1}$$

$$\left[\alpha^2,\, \beta^2,\, \gamma^2\right] = \left(\frac{R}{2\,\rho}\right)^2 \times \left[a\,(b+c-a)\,,\, b\,(c+a-b)\,,\, c\,(a+b-c)\right]$$

Similar to the intouch triangle ($k^2 = R^3 \div 2\rho^3$) and to the excentral triangle ($k^2 = R \div 8\rho$). We

have the following central correspondances :

$\mathcal{T}_*$	$\mathcal{T}$	$\mathcal{T}_*$	$\mathcal{T}$	$\mathcal{T}_*$	$\mathcal{T}$	$\mathcal{T}_*$	$\mathcal{T}$
2	3817	133	121	542	2801	2393	527
3	946	134	122	647	3835	2501	4885
4	10	135	123	690	3887	2574	3308
5	5	136	124	804	926	2575	3307
6	142	137	125	924	522	2679	1566
20	4301	138	126	974	1387	2777	2802
25	3452	139	127	1112	3035	2781	528
30	517	143	140	1154	30	2782	2808
39	2140	184	226	1205	3254	2790	2810
51	2	185	1	1495	908	2794	2809
52	3	235	1329	1503	518	2797	2821
53	141	389	1125	1510	523	2799	2820
65	178	403	3814	1531	1512	2848	2832
113	119	418	2051	1562	4904	3258	3259
114	118	427	2886	1568	1532	3564	971
115	116	428	3740	1596	3820	3566	3900
125	11	511	516	1637	4928	3574	442
128	113	512	514	1824	2090	3575	960
129	114	520	3667	1843	9	3917	1699
130	115	523	513	1986	214		
131	117	525	3309	1990	3834		
132	120	526	900	2052	3840		

For example, orthocenter $X(4, \mathcal{T}_*)$ is Spieker center $X(10, \mathcal{T})$.

Proof. Straightforward computations. In fact, $A'A^*$ and B^*C^* are orthogonal and the orthic triangle of $\mathcal{T}_*$ is the medial triangle of $\mathcal{T}$.

□

		$\phi_{\mathcal{T}}$ when ϕ is the isogonal conjugacy
medial	$\mathcal{C}_2$	$(v+w-u)\left((u+v-w)\,b^2+(u-v+w)\,c^2\right)$
antimedial	$\mathcal{A}_2$	$\dfrac{-a^2}{v+w}+\dfrac{b^2}{u+w}+\dfrac{c^2}{u+v}$
orthic	$\mathcal{C}_4$	$u\,(-S_a u+S_b v+S_c w)$
tangential	$\mathcal{A}_6$	$a^2\left(\dfrac{-a^2}{c^2v+wb^2}+\dfrac{b^2}{uc^2+wa^2}+\dfrac{c^2}{ub^2+a^2v}\right)$
excentral	$\mathcal{A}_1$	$a\left(\frac{-1}{(b+c-a)(cv+bw)}+\frac{1}{(a-b+c)(cu+aw)}+\frac{1}{(b+a-c)(bu+av)}\right)$

		circumcircle of $\mathcal{T}$
medial	$\mathcal{C}_2$	$\sum\left(b^2+c^2-a^2\right)x^2-2\sum a^2yz$
antimedial	$\mathcal{A}_2$	$\sum a^2x^2+\left(a^2+b^2+c^2\right)\sum yz$
orthic	$\mathcal{C}_4$	$\sum\left(b^2+c^2-a^2\right)x^2-2\sum a^2yz$
tangential	$\mathcal{A}_6$	$a^2b^2c^2\sum\left(b^2+c^2-a^2\right)x^2+\left(\sum_6 a^4b^2-\sum_3 a^6\right)\sum a^2yz$
excentral	$\mathcal{A}_1$	$\sum bcx^2+(a+b+c)\sum ayz$

Figure 17.1: Special Triangles

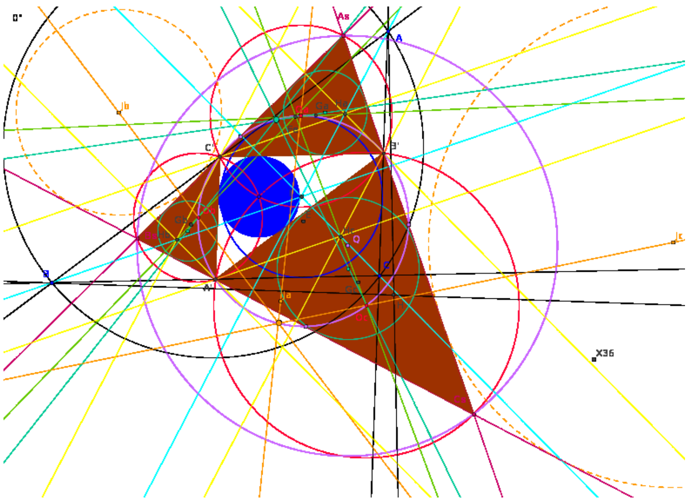


FIGURE 17.2: The star triangle

Chapter 18

Binary operations

18.1 Complementary and anticomplementary conjugates

Definition 18.1.1. comcon, anticomcon. For points $P = p : q : r$ and $U = u : v : w$, neither lying on a sideline of ABC, the P-complementary and P-anticomplementary conjugates of U are defined as in Figure 18.1, where k is the P-isoconjkim. Most of the time, $P = X_1$ and isoconjkim reduces to isogonal conjugacy.

18.2 cevapoint, cevaconjugate, crosspoint, crossconjugate

This subsection is provided to keep track of the names used in Kimberling ETC to define operations related to cevian nests.

Remark 18.2.1. The cevamul operation (aka cevapoint) has been defined in Section 3.8 by :

$$\begin{aligned}&\mathit{cevamul}\,(u : v : w,\ x : y : z) = \\ &\qquad\qquad (uz + wx)(uy + vx) : (vz + wy)\,(uy + vx) : (vz + wy)\,(uz + wx)\end{aligned}$$

This operation is clearly commutative and type-keeping. It's converse (when one of U, X) is fixed, i.e. $X = \mathit{cevadiv}\,(P, U)$ –aka "X is the P-cevaconjugate of U"– when $P = \mathit{cevamul}\,(U, X)$ is therefore a type-keeping (P, U)-map and an involutive U-map.

Remark 18.2.2. The crossmul operation (aka crosspoint) has been defined in Section 3.7 by :

$$\mathit{crossmul}\,(u : v : w,\ x : y : z) = (v\,z + w\,y)\,u\,x : (u\,z + w\,x)\,v\,y : (u\,y + v\,x)\,w\,z$$

This operation is clearly commutative and type-keeping. It's converse (when one of U, X) is fixed, i.e. $X = \mathit{crossdiv}\,(P, U)$ –aka "X is the P-crossconjugate of U"– when $P = \mathit{crossmul}\,(U, X)$ is therefore a type-keeping (P, U)-map and an involutive U-map.

$$\begin{tikzcd}
V \arrow[d, dotted, "\mathit{anticomcon}"] & X \arrow[l, "\mathit{anticompl}"'] \arrow[r, "\mathit{complem}"] \arrow[d, "k_P"] & U \arrow[d, dotted, "\mathit{comcon}"] \\
V' & X' \arrow[l, "\mathit{anticompl}"] \arrow[r, "\mathit{complem}"'] & U
\end{tikzcd}$$

$$x_{comcon} = \frac{b^3}{q(u - v + w)} + \frac{c^3}{r(u + v - w)}$$

$$x_{anticomcon} = -\frac{a^2}{p(v + w)} + \frac{b^3}{q(u + w)} + \frac{c^3}{r(u + v)}$$

Figure 18.1: The complementary and anticomplementary conjugates

Remark 18.2.3. As a comparison of $cevamul(P,U)$ and $crossmul(P,U)$, note that their barycentrics can be written as

$$\frac{1}{qw+rv}:\frac{1}{ru+pw}:\frac{1}{pv+qu}\qquad\qquad \frac{1}{qw}+\frac{1}{rv}:\frac{1}{ru}+\frac{1}{pw}:\frac{1}{pv}+\frac{1}{qu}$$

Properties of cross conjugates arise from those of ceva conjugates since :

$$crossdiv\,(P,U)*_b cevadiv\,(U,P)=P*_b U$$

18.3 crossdiff, crosssum, crosssumbar

Definition 18.3.1. crossdiff. In ETC, the crossdiff operator of two points $U=u:v:w$ and $X=x:y:z$ that aren't lying on a sideline of ABC has been defined, using barycentrics, as :

$$\mathrm{crossdiff}\,(U,X)=a^2\,(vz-wy):b^2\,(wx-uz):c^2\,(uy-vx)$$

Definition 18.3.2. crosssum. In ETC, the crosssum[1] of U and X that aren't lying on a sideline of ABC has been defined as :

$$\mathrm{crosssum}\,(U,X)=a^2\,(wy+vz):b^2\,(uz+wx):c^2\,(vx+uy)$$

with example X(1631) = crossum (X(116), X(513)).

Remark 18.3.3. Both operations are commutative. Defined that way, they are type-crossing, and are providing a line when entries are points.

Proposition 18.3.4. *The crosssum is constructible as* $\mathrm{crossmul}\,(isogon\,(U)\,,isogon\,(X))$ *or as* $isogon\,(cevamul\,(U,X))$. *Therefore a better definition is the globally type-keeping function :*

$$\mathrm{crosssum}_F\,(U,X)=f^2\,(wy+vz):g^2\,(uz+wx):h^2\,(vx+uy)$$

where $F=f:g:h$ *is either fixed point of the conjugacy.*

Remark 18.3.5. Defined that way, $\mathrm{crosssum}\,(U,X)$ is really different from the polar line of X wrt the circumconic $CC\,(U)\,:\,u\,yz+v\,zx+w\,xy=0$ since this line is the next coming crosssumbar.

Proposition 18.3.6. *The crosssumbar of two points* $U=u:v:w$ *and* $X=x:y:z$*, neither lying on a sideline of* ABC *is defined as the polar line of point* X *wrt circumconic* $CC\,(U)$*. Then* $crosssumbar\,(U,U)=isotom\,(U)$ *and*

$$\begin{aligned}crosssumbar\,(U,X)&=complem\,(X\div_b U)\div_b U\\&=\mathrm{crossmul}(isot(U),\,isot(X))=isot\,(cevamul\,(U,\,X))\\&=(wy+vz):(uz+wx):(vx+uy)\end{aligned}$$

Operation crosssumbar is commutative and type-crossing (i.e output is a line when entries are points). It can be reverted using operation crosssumdiv defined by :

$$\begin{aligned}crosssumdiv(P,\,U)&=cevadiv(isot(P),\,U)\\&=(qv+rw-pu)\,u:(rw+pu-qv)\,v:(pu+qv-rw)\,w\end{aligned}$$

that has the right variance when P *is a line and* U *is a point.*

18.4 Hirstpoint aka Hirst inverse

Definition 18.4.1. Hirstpoint. Suppose $P=p:q:r$ and $U=u:v:w$ are distinct points, neither lying on a sideline of ABC. The hirstpoint X is the point of intersection of the line PU and the polar of U with respect to the circumconic $CC\,(P)$ conic :

[1] Regarding the neologism "crosssum" placed here on 5/28/03, what words in the English language have spellings containing three consecutive identical letters?

$$p\,yz + q\,zx + r\,xy = 0.$$

Proposition 18.4.2. *We have the following properties :*
(i) H is a type-keeping operation as a "ramified product" of type-crossing transforms and :

$$\begin{aligned} hirstpoint\,(P, U) &= (P \wedge U) \wedge crosssumbar\,(P, U) \\ &= u^2\,q\,r - p^2\,v\,w : p\,v^2\,r - u\,q^2\,w : p\,q\,w^2 - u\,v\,r^2 \end{aligned} \tag{18.1}$$

(ii) H is commutative from the duality properties of polarization.
(iii) $H\,(P, U) = 0 : 0 : 0$ occurs only when $U = P$
(iv) $H\,(P, U) = P$ if and only if U lies on the polar line of P
(v) $H\,(P,\ H\,(P, U))$ is either $0 : 0 : 0$ or U. Indeterminate form is obtained (a) on the polar line of P and (b) on a conic containing P and having P has perspector ... i.e. a conic whose only real point is P.

Proof. Direct inspection for all properties. To be precise, these properties are valid only on the real part of the world. For example, (i) gives $U = p : jq : j^2 r$ where $j^3 = 1$, i.e. $U = P$ and two other "imagined" solutions. □

All these properties show that "Hirst inverse" is a poorly chosen term, since we aren't dividing, but multiplying. Concerning the designation "Hirst inverse," see the contribution Gunter Weiss: http://mathforum.org/kb/message.jspa?messageID=1178474.

18.5 Line conjugate

Suppose $P = p : q : r$ and $U = u : v : w$ are distinct points, neither equal to A, B, or C. The P-line conjugate of U is the point whose trilinears are given by :

$$p(v^2 + w^2) - u(qv + rw) : q(w^2 + u^2) - v(rw + pu) : r(u^2 + v^2) - w(pu + qv)$$

This is the point of intersection of line PU and the tripolar of the isogonal conjugate of U.

Using the same formula with barycentrics, another point is obtained, that is the intersection of PU and the dual of U. So what ?

Chapter 19

Two transforms

19.1 Collings transform

Lemma 19.1.1. *Let M_i, $1 \le i \le 5$, be five (different) points, not four of them on the same line, such that midpoint $(M_1, M_2) =$ midpoint $(M_3, M_4) = P$. They determine uniquely a conic whose center is P and contains $reflection(P, M_5)$.*

Proof. Take P as origin of the Euclidean coordinate system and consider determinant γ whose lines are $\left[x_i^2, x_i y_i, y_i^2, x_i, y_i, 1\right]$, the last line ($i = 6$) referring to the generic point of the plane. Since $x_2 = -x_1$, $\cdots$ γ can be factored into $(x_1 y_3 - x_3 y_1)$ times an expression without terms of first degree in x_6, y_6. $\square$

Lemma 19.1.2. *Let M_i, $1 \le i \le 4$, be four different points, not on the same line. The locus :*

$$\Gamma = \{\mathit{center}\,(\gamma) \mid \gamma \text{ is a conic and } M_1, M_2, M_3, M_4 \in \gamma\}$$

is a conic. It contains the six midpoint(M_i, M_j) and its center is $K = \sum M_i/4$.

Proof. Γ is a conic since degree is two. When $P = \mathit{midpoint}\,(M_1, M_2)$, the preceding lemma can be applied to $M_1, M_2, M_3, \mathit{reflection}\,(P, M_3)\,, M_4$, defining a conic whose center is P, so that $P \in \Gamma$. Now, the lemma can be applied to Γ itself, since $K = \mathit{midpoint}\,(\mathit{midpoint}\,(M_1, M_2)\,, \mathit{midpoint}\,(M_3, M_4))$ –and cyclically. $\square$

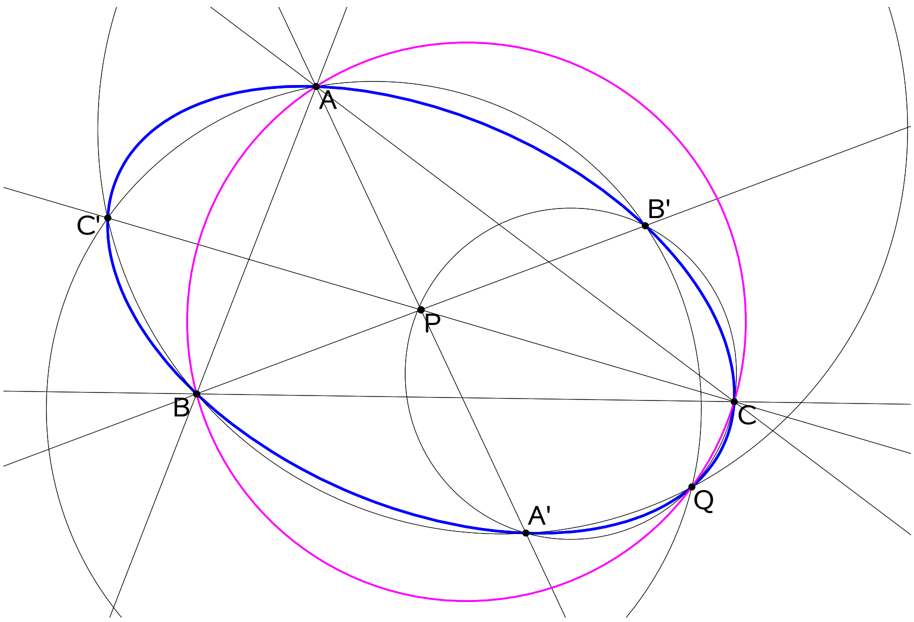


Figure 19.1: Collings configuration

Proposition 19.1.3. *Let P be a point not on a sideline of ABC, and A', B', C' the reflections of A, B, C in P.*
(i) It exists a conic γ through the six points A, B, C, A', B', C', its center is P and its perspector is $U = P *_b \text{anticomplem}(P)$, so that $\gamma = CC(U)$. This conic intersects the circumcircle at point $Q = \text{isotom}(X_6 \wedge U)$, i.e. the tripole of line UX_6. Moreover, the circumcircles of triangles $AB'C'$, $A'BC'$ and $A'B'C$ are also passing through point Q.
(ii) Conversely, when Q is given on the circumcircle, the locus of P is $\text{conicev}(X_2, Q)$. This conic goes through the three $AB \cap CQ$ points, the three $\text{midpoint}(A, Q)$ and through four fixed points : the vertices of medial triangle and through the circumcenter X_3. Moreover, this conic is a rectangular hyperbola, its center is $K = (A + B + C + Q)/4$ and belongs to the nine points circle of the medial triangle.
(iii) The anticomplement of this RH is the rectangular ABC-circumhyperbola whose center is the complement of Q.

Proof. For (i), only Q belongs to circumcircle of $AB'C'$ has to be proved. Barycentric computation. For (ii), point $AB \cap CQ$ lead to the degenerate conic $AB \cup CQ$ (and cyclically) while the six midpoints come from the lemma. When $P = X_3$, conic γ is the circumcircle... and passes through A, B, C, Q and $X_3 \in \Gamma$. But X_3 of ABC is X_4 of the medial triangle, and Γ contains an orthic configuration, characteristic property of a rectangular hyperbola. □

For the sake of exhaustivity, if barycentrics of P are $p : q : r$ then barycentrics of Q are

$$Q_x : Q_y : Q_y \quad where \quad Q_x = \frac{1}{r(p+q-r)b^2 - q(p-q+r)c^2}$$

(and cyclically in a, b, c and p, q, r too). The transformation $P \mapsto Q$ was described by Collings (1974) and was further discussed by (Grinberg, 2003b).

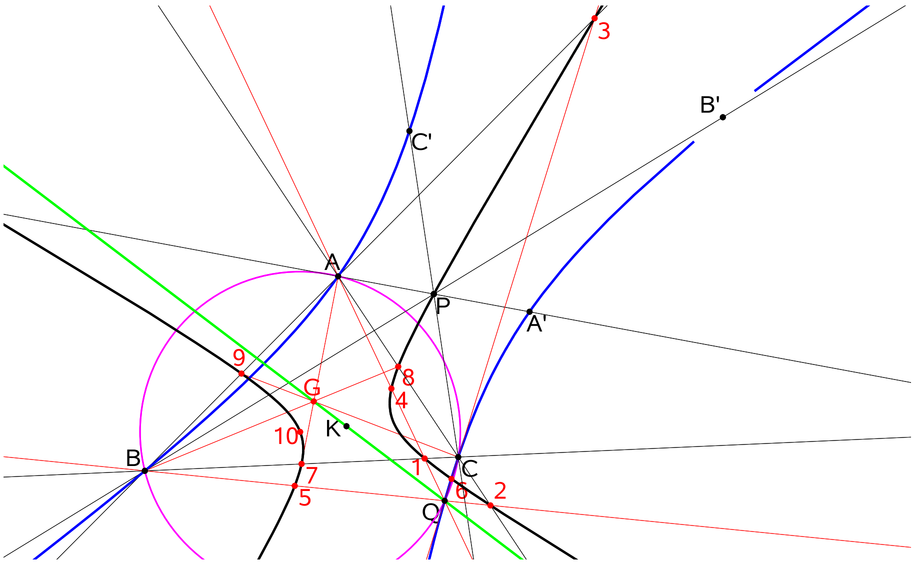


Figure 19.2: Collings locus is a ten points rectangular hyperbola

Example 19.1.4. Examples are as follows :

Q	points on the conic	wrt medial triangle
X_{74}	125	
X_{98}	115, 868	
X_{99}	2, 39, 114, 618, 619, 629, 630, 641, 642, 1125	Kiepert hyperbola
X_{100}	1, 9, 10, 119, 142, 214, 442, 1145	Feuerbach hyperbola
X_{107}	4, 133, 800, 1249	
X_{110}	5, 6, 113, 141, 206, 942, 960, 1147, 1209	Jerabek hyperbola
X_{476}	30	

19.2 Brisse Transform

From http://en.wikipedia.org/wiki/Poncelet's_porism :

> In geometry, Poncelet's porism, named after French engineer and mathematician Jean-Victor Poncelet, states the following: Let C and D be two plane conics. If it is possible to find, for a given $n > 2$, one n-sided polygon which is simultaneously inscribed in C and circumscribed around D, then it is possible to find infinitely many of them.
>
> Poncelet's porism can be proved via elliptic curves; geometrically this depends on the representation of an elliptic curve as the double cover of C with four ramification points. (Note that C is isomorphic to the projective line.) The relevant ramification is over the four points of C where the conics intersect. (There are four such points by Bezout's theorem). One can also describe the elliptic curve as a double cover of D; in this case, the ramification is over the contact points of the four bitangents.

Applied to a point $P = p : q : r$ on the circumcircle Γ of triangle ABC, this gives the existence of points P_2, P_3 on the circumcircle such that the incircle of ABC is also the incircle of PP_2P_3. As given in Brisse (2001), barycentric equation of P_2P_3 is :

$$\frac{(-a+b+c)p}{a^2}\,x + \frac{(a-b+c)q}{b^2}\,y + \frac{(a+b-c)r}{c^2}\,z = 0$$

and the contact point of this line with the incircle is given by the (barycentric) Brice transform :

$$T\left(P\right) = \frac{a^4}{(b+c-a)p^2} : \frac{b^4}{(c+a-b)q^2} : \frac{c^4}{(a+b-c)r^2}.$$

The converse transform, applied to a point $U = u : v : w$ on the incircle, is :

$$T^{-1}\left(U\right) = \frac{a^2}{\sqrt{u}\,\sqrt{-a+b+c}} : \frac{b^2}{\sqrt{v}\,\sqrt{a-b+c}} : \frac{c^2}{\sqrt{w}\,\sqrt{a+b-c}}$$

where all involved quantities are non-negative.

Examples:$X(11)$ = Feuerbach point = $T(X(109))$, $X(1317)$ = incircle-antipode of $X(11)$ = $T(X(106))$.

Exercise 19.2.1. Still open is the question posed in (Yiu, 2003) : list *all* polynomial centers on the incircle having low degree and prove that there are no others. Here, "degree" of $X = p(a,b,c) : p(b,c,a) : p(c,a,b)$ [barycentrics] refers to the degree of homogeneity of $p(a,b,c)$, and "low" means less than 6. For example, the Feuerbach point, $X(11)$, has degree 3.

Chapter 20

Combos

20.1 A new feature : combos

20.1.1 About combos

Definition 20.1.1. Combos (November 1, 2011). Suppose that P and U are finite points having normalized barycentric coordinates (p,q,r) and (u,v,w). (Normalized means that p + q + r = 1 and u + v + w = 1.) Suppose that f = f(a,b,c) and g = g(a,b,c) are nonzero homogeneous functions having the same degree of homogeneity. Let x = fp + gu, y = fq + gv, z = fr + gw. The (f,g) combo of P and U, denoted by f*P + g*U, is introduced here as the point X = x : y : z (homogeneous barycentric coordinates); the normalized barycentric coordinates of X are (kx,ky,kz), where k=1/(x+y+z).

Remark 20.1.2. Note 1. If P and U are given by normalized trilinear coordinates (instead of barycentric), then f*P + g*U has homogeneous trilinears fp+gu : fq+gv : fr+gw, which is symbolically identical to the homogenous barycentrics for f*P + g*U. The normalized trilinear coordinates for X are (hx,hy,hz), where h=2*area(ABC))/(ax + by + cz).

Remark 20.1.3. Note 2. The definition of combo readily extends to finite sets of finite points. In particular, the (f,g,h) combo of P = (p,q,r), U = (u,v,w), J = (j,k,m) is given by fp + gu + hj : fq + gv + hk : fr + gw + hm and denoted by f*P + g*U + h*J.

Remark 20.1.4. Note 3. f*P + g*U is collinear with P and U, and its {P,Q}-harmonic conjugate is fp - gu : fq - gv : fr - gw.

Remark 20.1.5. Note 4. Suppose that f,g,h are homogeneous symmetric functions all of the same degree of homogeneity, and suppose that X, X', X" are triangle centers. Then f*X + g*X' + h*X" is a triangle center.

Remark 20.1.6. Note 5. Suppose that X, X', X", X"' are triangle centers and X', X", X"' are not collinear. Then there exist f,g,h as in Note 4 such that X"' = f*X + g*X' + h*X". That is, loosely speaking, every triangle center is a linear combination of any other three noncollinear triangle centers.

Remark 20.1.7. Note 6. Continuing from Note 5, examples of f,g,h are conveniently given using Conway symbols for a triangle ABC with sidelengths a,b,c. Conway symbols and certain classical symbols are identified here:

S = 2*area(ABC)
SA = (b2 + c2 - a2)/2 = bc cos A
SB = (c2 + a2 - b2)/2 = ca cos B
SC = (a2 + b2 - c2)/2 = ab cos C
s=(a+b+c)/2
sa = (b + c - a)/2
sb = (c + a - b)/2
sc = (a + b - c)/2
r = inradius = S/(a + b + c)
R = circumradius = abc/(2S)
cot(ω) = (a2 + b2 + c2)/(2S), where ω is the Brocard angle

Remark 20.1.8. Note 7. The definition of combo along with many examples were developed by Peter Moses prior to November 1, 2011. After that combos have been further developed by Peter Moses, Randy Hutson, and Clark Kimberling.

Examples of two-point combos:
X(175) = 2s*X(1) - (r + 4R)*X(7)
X(176) = 2s*X(1) + (r + 4R)*X(7)
X(481) = s*X(1) - (r + 4R)*X(7)
X(482) = s*X(1) + (r + 4R)*X(7)
Examples of three-point combos: see below at X(1), X(2), etc.

Remark 20.1.9. Note 8. Suppose that T is a (central) triangle with vertices A',B',C' given by normalize barycentrics. Then T is represented by a 3x3 matrix with row sums equal to 1. Let NT denote the set of these matrices and let * denote matrix multiplication. Then NT is closed under *. Also, NT is closed under matrix inversion, so that (NT, *) is a group. Once normalized, any central T can be used to produce triangle centers as combos of the form Xcom(nT); see the preambles to X(3663) and X(3739).

The discussion of combos near the beginning of ETC is continued here. Suppose that T is a central triangle, and let nT its normalization, so that the triangle nT is essentially a 3x3 matrix with row sums equal to 1, and the rows of nT are normalized barycentrics for the A-, B-, C- vertices of T. Let X be a triangle center, given by barycentrics x : y : z, not necessarily normalized. The point whose rows are the matrix product X*(nT) is then a triangle center, denoted by Xcom(T).

Among central triangles T are cevian and anticevian triangles and others described at MathWorld. A brief list follows, with A-vertices given in barycentrics (not normalized):
Intouch triangle = cevian triangle of X(7) A-vertex = 0 : 1/(c + a - b) : 1/(a + b - c)
Extouch triangle = cevian triangle of X(8) A-vertex = 0 : c + a - b : a + b - c
Incentral triangle = cevian triangle of X(1) A-vertex = 0 : b : c
Excentral triangle = anticevian triangle of X(1) A-vertex = -a : b : c
Hexyl triangle A-vertex = a(1 + a1 + b1 + c1) : b(-1 + a1 + b1 - c1) : c(1 + a1 - b1 + c1), where a1 = cos A, b1 = cos B, c1 = cos C
Tangential triangle = anticevian triangle of X(6) A-vertex = -a2 : b2 : c2
1st Brocard triangle A-vertex = a : c2 : b2

20.1.2 More combos

Continuing the discussion of points Xcom(T), suppose that T is an arbitrary triangle, and let nT denote the normalization of T. Let NT denote the set of these triangles, as matrices, and let * denote matrix multiplication. Then NT is closed under *. Also, NT is closed under matrix inversion. Consequently, (NT, *) is a group, comparable to the group of stochastic matrices.

Suppose that T1 and T2 are triangles. In many cases, the product T1*T2 is well-defined (e.g., TCCT, page 175). However, n(T1*T2) may not be n(T1)*n(T2) if T1 and T2 are not normalized. Therefore, it is important, when dealing with products, to include the "n" if it is intended.

As a class of examples of triangles defined by matrix products, suppose that T1 is the cevian triangle of a triangle center f : g : h (barycentrics) and that T2 is the cevian triangle of a triangle center u : v : w. The A-vertex of the triangle T3 = (nT1)*(nT2) is given by
u(gu + hu + gv +hw) : hv(u + w) : gw(u + v),
from which it can be shown that T3 is perspective to the triangle ABC, with perspector
P = ghu(v + w) : hfv(w + u) : fgw(u + v).
Also, T3 is perspective to T2, with perspector
Q = u(v + w)(gu + hu + gv + hw) : v(w + u)(hv + fv + hw + fu) : w(u + v)(fw + gw + fu + gv).

Thus, T3 can be constructed directly from the pairs {P, ABC} and {Q, T2}. Regarding the possibility that T3 is also perspective to T1, the concurrence determinant for this condition factors as F1F2F3, where
F1 = u + v + w, F2 = fu(g + h) + gv(h + f) + hw(f + g), F3 = (fghuvw)2[(gu)-2 - (hu)-2 + (hv)-2 - (fv)-2 + (fw)-2 - (gw)-2].

Consequently, the perspectivity holds if and only if one of the three factors is 0, and from this result, various geometric results can be obtained.

Example: The Incentral and Medial triangles result by taking f : g : h = a : b : c and u : v : w = 1 : 1: 1. The product n(Incentral)*n(Medial) has vertices

A' = b + c : c : b B' = c : c + a : a C' = b : a : a + b

and its inverse has A-vertex A" = bc + ca + ab : -bc - ca + ab : -bc + ca - ab, from which B" and C" are obtained cyclically.

The product n(Medial)*n(Incentral) has A-vertex a(2a + b + c) : b(c + a) : c(a + b), and the inverse of this product has A-vertex (a + b + c)(b + c) : -c(a + c) : -b(a + b).

Having considered the group NT of normalized triangles, we turn next to 3-point combos based on product triangles and inverse triangles. Recall that such a combo, denoted by Xcom(T), is defined, as in the preamble to X(3663), by the matrix product (x y z)*(nT), where nT is the normalization of T.

20.1.3 More-more combos

Many triangle centers can be defined (as just above) by the form Xcom(nT), where T denotes a central triangle. In order to present such centers, it is helpful to introduce the notation T(f(a,b,c), g(a,b,c)) for central triangles. Following TCCT, pages 53-54, suppose that each of f(a,b,c) and g(a,b,c) is a center-function or the zero function, and that one of these three conditions holds:

the degree of homogeneity of g equals that of f; f is the zero function and g is not the zero function; g is the zero function and f is not the zero function.

There are two cases to be considered: If g(a,b,c)=g(a,b,c), then the central triangle T(f(a,b,c), g(a,b,c)) is defined by the following 3x3 matrix (whose rows give homogeneous coordinates for the A-, B-, C- vertices, respectively):

f(a,b,c) g(b,c,a) g(c,a,b) g(a,b,c) f(b,c,a) g(c,a,b) g(a,b,c) g(b,c,a) f(c,a,b)

If g(a,b,c) is not equal to g(a,b,c), then the central triangle T(f(a,b,c),g(a,b,c)) is defined by the following 3x3 matrix:

f(a,b,c) g(b,c,a) g(c,b,a) g(a,c,b) f(b,c,a) g(c,a,b) g(a,b,c) g(b,a,c) f(c,a,b)

If the homogenous coordinates are chosen to be barycentric, as they are for present purposes, then the new notation for central triangles is typified by these examples:

Medial triangle = T(1, 0); Anticomplementary triangle = T(-1, 1) Incentral triangle = T(a, 0); Excentral triangle = T(-a, b) Euler triangle = T(2(b4 + c4 - b2c2 - c2a2 - a2b2), (b2 + c2 - a2)(a2 + b2 - c2)) 2nd Euler triangle = T(2a2(b4 + c4 - b2c2 - c2a2 - a2b2), (b2 - c2 - a2)(a4 + b4 + c4 - 2a2c2 - 2b2c2)) 3rd Euler triangle = T((b - c)2, c2 - a2 - bc) 4th Euler triangle = T((b + c)2, c2 - a2 + bc) 5th Euler triangle = T(2(b2 + c2), b2 + c2 - a2) (The vertices of the five Euler triangles lie on the nine-point circle.)

The notation F(f(a,b,c),g(a,b,c)) can be used to define several more triangles; in each case, two of the perspectivities can be used to construct the triangle.

T(bc, b2) is perspective to ABC with perspector X(6); other such pairs: incentral, X(192), excentral, X(1045); tangential, X(6); anticomplementary, X(192)

T(-bc, b2) is perspective to ABC with perspector X(6); incentral, X(2); tangential, X(6); T(bc,b2), X(6)

T(bc, c2) is perspective to ABC with perspector X(76); anticomplementary, X(8); 4th Brocard, X(76), Fuhrmann, X(8); cevian(X(350)),X(1)

T(-bc, c2) is perspective to ABC with perspector X(76); cevian(X(321)),X(75); anticevian(X(8)), X(2)

T(a2, a2 - b2) is perspective to the medial triangle with perspector X(3); tangential , X(3); symmedial, X(2); cevian(X(287)), X(98)

T(a2, a2 - c2) is perspective to the tangential triangle with perspector X(25); orthic , X(25); medial, X(6); 1st Lemoine, X(1383); circummedial, X(251)

T(a2, a2 + b2) is perspective to ABC with perspector X(83); medial, X(6); 1st Neuberg, X(182)

T(a2, a2 + c2) is perspective to ABC with perspector X(141); medial, X(3); 1st Neuberg, X(182); T(a2,a2 - c2), X(2); cevian(X(69)), X(6); cevian(X(76)), X(2); cevian(X(3313)),X(39)

The triangles IT1 and IT2 are triply perspective to ABC (as are T1 and T3). Order-label IT1 as A'B'C'. The three perspectivities are then given by

AA'⋂BB'⋂CC' = X(3862) AB'⋂BC'⋂CA' = f(a,b,c) : f(b,c,a) : f(c,a,b), where f(a,b,c) = ab(a2 - bc)(c2 - ab)(b2 + bc + c2) AC'⋂BA'⋂CB' = f(a,c,b) : f(b,a,c) : f(c,b,a)

Peter Moses noted (12/23/2011) that T(bc, b2) is triply perspective to ABC. With T(bc, b2) order-labeled as A'B'C', the three perspectivities with perspectors are as follows:

AA'⋂BB'⋂CC' = X(6); AB'⋂BC'⋂CA' = P(6); AC'⋂BA'⋂CB' = U(6). (The notation P(k) and U(k) refers to a bicentric pair of points; see More at the top of ETC.) Likewise, T(bc, c2) is

triply perspective to ABC, with perspectors: X(76), P(10), and U(10).

(This section was added to ETC on 12/23/2011.)

20.1.4 More-more-more combos

The Euler triangle and the 2nd, 3rd, 4th, and 5th Euler triangles are discussed in the preamble to X(3758), along with eight other central triangles using the notation T(f(a,b,c), g(a,b,c)). Recall that the inverse of a normalized central triangle T, denoted by Inverse(nT) or Inverse(n(T)), is also a normalized central triangle. Barycentrics for the inverse of each of the 13 triangles and properties of these triangles are given (Peter Moses, December 2011) as follows:

Inverse(n(Euler triangle)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = (b2 - c2)2 + a2(3a2 - 4b2 - 4c2) g(a,b,c) = b4 - (c2 - a2)2

Inverse(n(2nd Euler triangle)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = a2(a2 - b2 - c2)(a6 + b6 + c6 - a4b2 - a4c2 - a2b4 - a2c4 - 2a2b2c2 - b4c2 - b2c4) g(a,b,c) = b2(b2 - c2 - a2(a6 + b6 - c6 - a4b2 - 3a4c2 - a2b4 + 3a2c4 + 2a2b2c2 - 3b4c2 + 3b2c4)

Inverse(n(3rd Euler triangle)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = a(b2 + c2 - ab - ac + bc) g(a,b,c) = b(a2 - ab - ac + bc - c2)

Inverse(n(4th Euler triangle)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = a(a - b - c)(b2 + c2 + ab + ac) g(a,b,c) = b(-a + b + c)(a2 + ab - ac - bc - c2)

Inverse(n(5th Euler triangle)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = (a2 - 2b2 - 2c2)(3a2 + b2 + c2) g(a,b,c) = -(a2 - b2 - c2)(2a2 - b2 + 2c2)

Inverse(n(T(bc,b2)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = -bc(a2 - bc)(b2 + c2 + bc) g(a,b,c) = b2(ab - c2)(a2 + c2 + ac)

Inverse(n(T(-bc,b2)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = bc(a2 - bc)(b2 + c2 - bc) g(a,b,c) = b2(ab + c2)(a2 + c2 - ac)

Inverse(n(T(bc,c2)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = -a(a2 - bc)(b2 + c2 + bc) g(a,b,c) = (ab - c2)(a2 + c2 + ac)

Inverse(n(T(-bc,c2)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = -a(a2 - bc)(b2 + c2 - bc) g(a,b,c) = (ab + c2)(a2 + c2 - ac)

Inverse(n(T(a2,a2 - b2)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = (3a2 - b2 - c2)(b4 + c4 - b2c2) g(a,b,c) = (a2 - 3b2 + c2)(a2b2 - 2b2c2 + c4)

Inverse(n(T(a2,a2 - c2)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = a2(b2 + c2 - 3a2) g(a,b,c) =(a2 - c2)(a2 - 3b2 + c2)

Inverse(n(T(a2,a2 + b2)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = -(3a2 + b2 + c2)(b4 + c4 + b2c2) g(a,b,c) = (a2 + 3b2 + c2)(a2b2 + c4)

Inverse(n(T(a2,a2 + c2)) = T(f(a,b,c), g(a,b,c)), where f(a,b,c) = -a2(a2 + b2 + c2)(3a2 + b2 + c2) g(a,b,c) = (a2 + c2)(a2 + b2 - c2)(a2 + 3b2 + c2)

For present purposes, label these last triangles as IT1, IT2, IT3, IT4, IT5, IT6, IT7, IT8; then all except IT5 and IT6 are perspective to the reference triangle ABC; perspectors are described at X(3862)-X(3866). Label the corresponding original triangles T1, T2, T3, T4, T5, T6, T7, T8; then the following pairs are perspective: (T1, IT1), (T2, IT2), (T3,IT3), (T4, IT4), (T7, IT7), (T8,IT8); coordinates for the perspectors are long and omitted.

The triangles IT1 and IT2 are triply perspective to ABC (as are T1 and T3). Order-label IT1 as A'B'C'. The three perspectivities are then given by

AA'$\bigcap$BB'$\bigcap$CC' = X(3862) AB'$\bigcap$BC'$\bigcap$CA' = f(a,b,c) : f(b,c,a) : f(c,a,b), where f(a,b,c) = ab(a2 - bc)(c2 - ab)(b2 + bc + c2) AC'$\bigcap$BA'$\bigcap$CB' = f(a,c,b) : f(b,a,c) : f(c,b,a)

Note that the second two perspectors are a bicentric pair (not triangle centers; see TCCT, page 47); likewise for the other triple perspectivity, with IT3 order-labeled as A'B'C':

AA'$\bigcap$BB'$\bigcap$CC' = X(3864) AB'$\bigcap$BC'$\bigcap$CA' = g(a,b,c) : g(b,c,a) : g(c,a,b), where g(a,b,c) = a(a2 - bc)(c2 - ab)(b2 + bc + c2) AC'$\bigcap$BA'$\bigcap$CB' = g(a,c,b) : g(b,a,c) : g(c,b,a)

underbar

20.1.5 more-more-more-more combos

(Pending: will tell constructions for the following triangles and their inverse normalizations) T(a, c)

T(-a, c)

T(a, b - c)

T(a, c - b)

T(a2, bc)
T(-a2, bc)
T(bc, bc + c2)
T(-bc, bc + c2)
T(bc, b2 + bc)
T(-bc, b2 + bc))

Chapter 21

Import the 3588-4994 part of ETC

21.1 First, recreate the database

1. Nowadays, ETC is made of two parts. Page one covers the 1-2000 part, and page two covers the 2001-4994 part. We use `create-data-file.bat` to split the web file in two parts:

   - the data part, selecting lines that start by `"<h3 id="`
   - the comment part, selecting the other lines

2. The data part contains a line per recorded point. The script `create-fiche` transcripts each line into the csv of a record, and then send it to the database. Since ETC is often hand-patched, the script is not so straightforward.

3. A web server is required. We are using apache2. Should start chrooted on boot.

4. A database server is required. We are using mlysql. Should start chrooted on boot.

5. Dare to create security holes ! Users mysql and wwwrun are chrooted and must remain chrooted ! Acceptable are links... when wwwrun is really chrooted and cannot use links to escape outside.

6. User is called `etcetc`. Privileges are: (1) everything on `etcetc` from localhost ; (2) FILE privilege from localhost.

7. Database is called `etcetc` and created empty. Then `center-09.sql.gz` is imported. This doesnot work with the uncompressed version (file too long). This creates the main table, called `center`. The purpose of this table is to be filled by records reformatted from the ETC web page.

8. Moreover, some repetitive tasks are better programmed than executed by random clicks. A local web page can launch some *.php scripts that create auxiliary tables or files, import files into temporary tables and commit some changes from auxiliary tables into the main table (`center`).

9. The global purpose is to maintain a *.cvs file that can be loaded each time Maple is launched. The corresponding table is called `bar_igtca` and contains :

Field	Type		objet
num	int	5	X(n)
ff	tinyint	1	1=at infinity
gg	smallint	6	isogon
tt	smallint	6	isotom
cc	smallint	6	complement
ac	smallint	6	anticomplem
bary1	varchar	500	barycentrics
dg	int	5	dg of lubin
lubin	varchar	500	lubin affix
x (6)	varchar	30	numeric coord
hc	varchar	30	search key
gerade	varchar	1500	the lines

10. Table and file `barz.csv` are used to rework the syntax of the barycentrics, until the string becomes parseable and gives an expression having the right searchkey.

21.2 Points related to Morley : 3272-3283

to be seen again

21.3 Some errors in the web page

3590 barycentrics are wrong (should be 6S, not 3S). Same apply to 3591-3595

3599 ending)

3611 ending)

3632 lines are unreadable and $a-2b-2c$ is 3679

3637 over 500 char ==> shortened using Conway symbols.

3638 ending)

3639 ending)

3641 starting (

3642 a squared point. Better written as $\left(2a^4-b^4-c^4+2b^2c^2-a^2b^2-a^2c^2\right)+4\sqrt{3}S(c^2+b^2)$

3646 searchkey is wrong. 2.80169029270208151552171148982

3648 should be $(4(2+3R/r))S^2-(ac+2S_b)(ab+2S_c)$. The other is also wrong (a factor $-a$ is missing).

3654 a parenthesis is missing before the last term. Should be $\cdots a^2b^2)-3k)$.

3657 trilinears are wrong

3658 should be

$$\frac{a\left(a^3b+a^3c-a^2b^2-a^2c^2-ab^3+ab^2c+abc^2-ac^3+b^4-2\,b^2c^2+c^4\right)}{b^2-c^2}$$

3660 should be $a\left(ba^2+ca^2+2abc-2ac^2-2ab^2+b^3+c^3-cb^2-bc^2\right)/(a-b-c)$

3677 initial html tag with an extra)

3684 should be $a(a^2-bc)(a-b-c)$

3697 should be $a(b+c)(a^2-b^2-4bc-c^2)$

remark : isotom(1292) a meme searchkey que 3617

3702 should be $bc(2a+b+c)(a-b-c)$. What is given is 4723

3706 should be $(a-b-c)(2bc+ab+ca)$. What is given is 3705

3724 should be $a^3(b+c)(a^2-b^2+bc-c^2)$. Sign of bc.

3738 should be $(b-c)(a-b-c)(a^2-b^2+bc-c^2)a$. A factor is missing.

3771 leading (is missing

3772 leading (is missing

3791 should be $2a^3+a^2b+a^2c-b^2c-bc^2$. What is given is 3790.

3793 should be $(3a^2+b^2+c^2)(2a^2-b^2-c^2)$

3800 should be $(b^2-c^2)(3a^2+b^2+c^2)$

3801 should be $(b^2-c^2)(b^2-bc+c^2)$ html syntax

3813 should be $(a^3-a^2b+a^2c-ab^2-4abc+b^3+b^2c)(a+b-c)$ some wrong exponents

3814 leading (

3816 should be $ab^2-4abc+ac^2-b^3+b^2c+bc^2-c^3$ some wrong exponents

3820 should be $a^2b^2+a^2c^2-4ab^2c-4abc^2-b^4+2b^2c^2-c^4$ sign

3825 should be $a^2b^2-2a^2bc+a^2c^2-ab^2c-abc^2-b^4+2b^2c^2-c^4$ some wrong exponents

3829 should be $3ab^2-4abc+3ac^2-3b^3+3b^2c+3bc^2-3c^3$ homogeneity

3839 missing term $5a^4+2a^2b^2+2a^2c^2-7b^4+14b^2c^2-7c^4$

3846 should be $a^2b+a^2c+3ab^2+8abc+3ac^2+4b^2c+4bc^2$. 3847 is given instead

3847 should be $2abc+b^3+c^3$. What is given looks like 3846

3853 should be $6a^4-a^2b^2-a^2c^2-5b^4+10b^2c^2-5c^4$. Leading 6 is missing

3866 should be $(3a^2+b^2+c^2)(a^2b^2+c^4)(a^2c^2+b^4)$

3867 should be $(3a^2+b^2+c^2)(b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)$

3879 should be $2a^2+ab+ac-b^2-c^2$

3894 should be $(2a^2b+2a^2c+3abc-2b^3-2c^3)a$. Homogeneity

3896 should be $(b+c)(2a^2-bc)$. What is given is 740.

3905 should be $ab^3-b^3c+a^4-bc^3+ac^3$

3949 should be $a(b+c)^2(a^2-b^2-c^2)$

3953 should be $a(b^3+ab^2-2abc+c^3+ac^2)$ symmetry

3960 should be $(b-c)(a^2-b^2+bc-c^2)a$

3964 should be $(a^2-b^2-c^2)^3a^2$

3966 should be $(a-b-c)*(a^2+a*b+a*c+b^2+c^2)$

3968 should be $(b+c)(a^2-b^2+7bc-c^2)a$. Symmetry

3984 should be $(a-3b-3c)(a^2-b^2-c^2)a$. Extra parenthesis.

4021	$2a^2+3ab+3ac+b^2-2bc+c^2$
4027	$(a^2-bc)^2(a^2+bc)^2$. What is given is 3801.
4036	$bc(b-c)(b+c)^2$
4044	$bc(b+c)(a^2-ab-ac-2bc)$. What is given is 4043.
4049	$(b-c)(b+c)/(2a-b-c)$
4050	$(a-b-c)(ab+ac-3bc)a$. Signum
4085	$(b+c)(2a^2+b^2-bc+c^2)$. Signum
4108	$(b^2-c^2)(2a^4+b^2c^2)$. missing exponents
4148	$(b-c)(a^2-bc)(a-b-c)^2$ missing power
4169	$(2a-b-c)(b+c)/(b-c)$ missing coeff 2
4317	missing leading (
4323	iterated //
4377	$(b+c)bc(2bc+a^2)$ Missing coeff, giving 4374
4382	repeated record
4383	repeated record
4401	$(b-c)(2a^2-bc)a$ Missing coeff, giving 659
4405	$4a^2-b^2-8bc-c^2$ Missing coeff
4407	extra ending)
4427	$(2a+b+c)/(b-c)$ Missing a
4433	$(b+c)(a-b-c)(a^2-bc)a$ What is given is 4087.
4435	$(b-c)(a-b-c)(a^2-bc)a$
4460	$5a^2+4ab+4ac-b^2-6bc-c^2$ What is given is 4440.
4466	missing opening (
4469	$a\,(b^2+bc+c^2)^2/(b+c)$ Missing exponent
4481	$a\,(b-c)(b^2+bc+c^2)/(b+c)$ What is given is 984
4490	$(b-c)(b^2+3bc+c^2)a$ what is given is 4489
4499	$4a^2+2ab+2ac+b^2+8bc+c^2$
4516	$(b-c)^2(b+c)(a-b-c)a$ What is given is 4041
4666	$(a^2-2ab-2ac+b^2-4bc+c^2)a$
4674	$a\,(b+c)/(2a-b-c)$ Divide, not multiply.
4750	$(b-c)(2a^2-b^2-c^2)$ Homogeneity
4756	$(a+2b+2c)/(b-c)$ Divide, not multiply. What is given is 10
4758	Extra ending)
4803	$(a-2b-2c)^2/(b+c)$ Missing power.
4820	$(b-c)(a+2b+2c)(a-b-c)$
4830	$(b-c)(3a+b+c)(a^2-bc)$ Extra power

4850 $(ab + ac + b^2 - bc + c^2)a$ What is given is 3681.

4878 $(b + c)(a^2 - 2ab - 2ac + b^2 + c^2)a^2$ What is given is 4440.

4902 $-3a^2 + ab + ac + 4b^2 - 8bc + 4c^2$ Provided result is non homogeneous.

4915 $(a - b - c)(a^2 - b^2 + 10bc - c^2)a$

4993 $(a^4 + a^2b^2 + a^2c^2 - 2b^4 + 4b^2c^2 - 2c^4)/(a^2b^2 + a^2c^2 - b^4 + 2b^2c^2 - c^4)$. No searchkey provided.

4994 $(a^4 - 3a^2b^2 - 3a^2c^2 + 2b^4 - 4b^2c^2 + 2c^4)/((a^2 - b^2 - c^2)(a^2b^2 + a^2c^2 - b^4 + 2b^2c^2 - c^4))$. No searchkey provided

Chapter 22

Specifications of the database

In this section, we give the specifications of our implementation of the Kimberling ETC. At the present moment, this implementation is restricted to a private use. The intent is to participate to a distributed maintenance of the database, together with providing "random search keys", specific to each local copy, in order to allow a distributed method of proof.

22.1 Encapsulating everything in *.php commands

1. Writing for eternity implies to keep track of each action. Writing explicit programs is the best way for that.

2. A file named `index.php` is somewhere in a file tree that is mapped into a web tree `~/public_html/etc`. This file contains a link to each command file. Database is somewhere else (in the protected mysql space `/var/lib/mysql/etcetc`).

3. An individual command file is as listed in Listing 22.1. The core of the file is the mysql line of command (2). When everything is written by *.php, the redirection is in line (4). Otherwise, the mysql messages are directed to the web page. When a post-treatment is required, it can be embedded in an external routine (6).

4. a list of items is build that way :
```
drop table if exists lesqui;
create table lesqui select 1 as xnum ;
insert into lesqui (xnum) values (829),(1105),(1288),...  ;
```

22.2 LyX/LaTeX

Don't forget the HAL line when using the Listing environment + prettyref + hyperref

```
\def\arraystretch{1.2}
\makeatletter
\floatstyle{boxed}
\restylefloat{algorithm}
\def\fname@algorithm{{\scshape Listing}}
\def\fnum@algorithm{\fname@algorithm~\thealgorithm\,\!}
\def\theHalgorithm{\thealgorithm}%%{ HELLO, I am HAL }
\makeatother
```

22.3 Structure

Evolving structure is as described in Table 22.1.

1. Today (2012/02/15), the fields num, ff, gg, tt, cc, ac, bary1, dg, lubin, xx,yy,zz,zxx,zyy,zyy, hc, gera are listed in `bar_igtca.csv`, and read each time Maple is launched.

```
1: <html> <head> <title> structure </title></head>
   <body> <h1><center> structure </center></h1>
     <a href="index.php">back to index</a> <br> <hr>
   <?php $ici="/home/douillet/public_html/etc";
   $cmd = <<<EOF

2:  use etcetc; describe center ;

3:  EOF;
   $fp = fopen("$ici/tmp_file.cmd", 'w');
     fwrite ($fp, $cmd); fclose($fp);
   $now = <<<EOF mysql -uetcetc -p****** -v --show-warnings

4:   < \$ici/tmp_file.cmd > $ici/tmp_structure.txt

5: EOF;
   echo "----------- mysql -------------<br>\n";
   echo $now ."<br><br>\n"; exec(\$now, $retour);
   $count = count($retour);
   for ($i = 0; $i < $count; $i++)
     { echo $retour[$i] . "<br>\n"; }
   echo "----------------------------------<br>\n";

6: system("$ici/.structure.bat", $retour);

7: echo "-structure is done-$retour---------------<br>\n";
   ?> </body> </html>
```

LISTING 22.1: The structure.php file

2. booleans : sf (special #6), mf (bare and Morley #33) , ff (infinity points #229)
3. liens gg (isogon), tt (isotom), cc (complem), ac (anticomplem), ho (orthopoint)
4. bary1 in strict Maple syntax, without spaces (there were remaining spaces in #79 of them, directly from the HTML file). Quantities S (area), S_a, S_b, S_c (the Conway symbols) are used. R is deprecated. Some radicals should be specially quoted.
5. dg (degree) and lubin (the Lubin affix), in strict Maple syntax.

22.3.1 other

Inverses in various circles : [iccir iincc inine ibroc iortc ibevc iflec islec] are circumcircle, incircle, npc, Brocard, Bevan, Lemoine1 et Lemoine2.

myorthj orthojoin of

mycocj complementary conjugate of

myaccj anticomplementary conjugate of

mycycc cyclocevian conjugate of

myreflex reflection of XI in XJ

mycrosc XI-cross conjugate of XJ

mycrosp crosspoint of XI and XJ

mycrosa crosssum of XI and XJ

mycrosd crossdifference of any

	Field	Type			Field	Type	
1	num	int	5	32	circa	smallint	6
2	name	varchar	60	33	cirna	smallint	6
3	ff	tinyint	1	34	orthj	smallint	6
4	gg	smallint	6	35	cocj	smallint	6
5	tt	smallint	6	36	accj	smallint	6
6	cc	smallint	6	37	cycc	smallint	6
7	ac	smallint	6	38	reflex	varchar	500
8	bary1	varchar	500	39	crosc	varchar	250
9	sf	tinyint	1	40	crosp	varchar	100
10	mf	tinyint	1	41	crosa	varchar	300
11	ho	smallint	6	42	crosd	varchar	100
12	bary2	varchar	400	43	hirst	varchar	150
13	bary3	varchar	100	44	alph	varchar	250
14	bary4	varchar	100	45	beth	varchar	250
15	baryf	varchar	10	46	midp	varchar	250
16	triliasis	mediumtext		47	ceva	varchar	250
17	baryasis	mediumtext		48	cevc	varchar	250
18	lg	smallint	6	49	licon	varchar	250
19	gerade	varchar	1500	50	eigct	varchar	250
20	curves	varchar	100	51	eigat	varchar	250
21	iccir	smallint	6	52	lam	varchar	50
22	iincc	smallint	6	53	psi	varchar	50
23	inine	smallint	6	54	refs	mediumtext	
24	ibroc	smallint	6	55	residuel	longtext	
25	iortc	smallint	6	56	xx	varchar	30
26	ibevc	smallint	6	57	yy	varchar	30
27	iflec	smallint	6	58	zz	varchar	30
28	islec	smallint	6	59	zxx	varchar	30
29	ifuhr	smallint	6	60	zyy	varchar	30
30				61	zzz	varchar	30
31				62	hc	varchar	30

Table 22.1: Structure of the database

myhirst XI-Hirst inverse of XJ

myalph XI-aleph conjugate of XJ

mybeth XI-beth conjugate of XJ

mymidp midpoint of XI and XJ

myceva cevapoint of XI and XJ

mycevc XI-Ceva conjugate of XJ

mylicon XI-line conjugate of XJ

myeigct eigencenter of cevian triangle of XI

myeigat eigencenter of anticevian triangle of XI

mypsi Ψ

mylam Λ

Chapter 23

Describing ETC

23.1 Various kinds of points

Table 23.1 describe the various species and subspecies that can be found in the Kimberling's database.

	species	subspecies		total	
	rational	a^2, b^2, c^2 only	784		
		a^2, b^2, c^2, S	241	#1025	
	second deg	a, b, c only	3593		
		a, b, c, S	90	#3683	
	surds	$\sqrt{2}$ (all a^2, S)	#8		
		$\sqrt{5}$ (all a^2, S)	#18		
		(among them, #12 $\sqrt{5+\sqrt{5}}$)			
		$\sqrt{3}$ (#125a^2S, #39 aS)	#164		
	radicals	$\sqrt{a\,(b+c-a)}$	#50		
		$\sqrt{(a+b+c) \div abc}$ used in $\cos\frac{A}{2}$ (#14)			
		$\sqrt{a^2b^2+\cdots}$	#66		
		$\sqrt{a^4-b^2c^2+\cdots}$	#62		
		$4S\,\lvert OH\rvert = \sqrt{a^6-b^4c^2\cdots+3a^2b^2c^2}$ Euler	#52		
		$\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ (useless ?)	#18		
		$\frac{\lvert OI\rvert}{R} = \sqrt{\frac{a^3-b^2c\cdots+3abc}{abc}}$ (mixed #2)	#12		
		$\sqrt{\rho^2+s^2} = \sqrt{\frac{a^2b\cdots+abc}{a+b+c}}$ Spieker circle	#8		
		$\sqrt{a^8-a^6b^2+a^4b^2c^2+\cdots}$ Euler-Steiner	#4		
		$\sqrt{2bc-a^2\cdots}$ Poncelet In/circum-circle	#2		
		dg10, Euler-Incircle	#2	#236	
	angles	$trig\,(A/3)$ etc	33		
		bare A	8		
		weird	2		
	special points	368-70, 1144, 3232	5	48	

Table 23.1: The various species of points in the Kimberling's database

23.2 Special points

The following six points have no explicit barycentrics, but are defined as the solution of some special equation.

23.2.1 X(368)

1. Situated on the second Brocard cubic, and on the anticomplement of the Kiepert hyperbola $CC\,(X_{523})$. Append $x+y+z=1$.
2. Substitute ency_ (=local solution). Eliminate x, y. Obtain a six degree equation in z, with $3z-1$ in factor (barycenter G).
3. Four of the roots are complex, and X_{368} is the remaining one.

23.2.2 X(369), X(3232) trisected perimeter points

There exist points A', B', C' on segments BC, CA, AB, respectively, such that :

$$AB' + AC' = BC' + BA' = CA' + CB' = (a+b+c)/3$$

Lines AA', BB', CC' concur in X(369). Yff found that barycentrics $p:q:r$ for X(369) can be obtained in terms of the unique real root, K, of the cubic polynomial equation :

$$2\,K^3 - 3\,(b+c+a)\,K^2 + \left(a^2+b^2+c^2+8\,(bc+ca+ab)\right)K \\ -\left(b^2c+c^2a+a^2b\right) - 5\left(bc^2+ca^2+ab^2\right) - 9\,abc = 0$$

and gives at the geometry conference held at Miami University of Ohio, 2004/10/02, a symmetric form for these barycentrics, namely :

$$p = K^2 - (2\,c+a)\,K - a^2 + b^2 + 2\,c^2 + 2\,bc + 3\,ac + 2\,ab$$

and cyclically. In the reference triangle, we have :

$$X_{369} = (0.4090242897, 0.3561467951, 0.2348289151)$$

On the other hand, there exist points A', B', C' on segments BC, CA, AB, respectively, such that $B'C + C'B = C'A + A'C = A'B + B'A = (a+b+c)/3$, and the lines AA', BB', CC' concur in X(3232), the isotomic conjugate of X(369).

23.2.3 X(370)

From http://pagesperso-orange.fr/bernard.gibert/Tables/table10.html, point X_{370} lies on conic :

$$(xy(a^2-b^2+c^2) + xz(a^2+b^2-c^2) + 2zya^2) - x(y+z+2x)\frac{abc}{R\sqrt{3}} = 0$$

and on the other two obtained cyclically. Two such conics have in common a vertex, X_{370} and other two (may be not real) points. In any case, X_{370} is inside triangle ABC.

23.2.4 X(1144)

Suppose P is a point inside triangle ABC. Let Sa be the square inscribed in triangle PBC, having two vertices on segment BC, one on PB, and one on PC. Define Sb and Sc cyclically. Then X_{1144} is the unique choice of P for which the three squares are congruent. $L(a,b,c)$ is the common length of the sides of the three squares. The function $L(a,b,c)$ is symmetric, homogeneous of degree 1, and satisfies $0 < L(a,\,b,\,c) < \min{(a,\,b,\,c)}$. L is the smallest root of :

$$\frac{a^2}{a-L} + \frac{b^2}{b-L} + \frac{c^2}{c-L} - 2\,\frac{S}{L} = 0$$

and point X_{1144} is obtained as :

$$\frac{a^2}{a-L} : \frac{b^2}{b-L} : \frac{c^2}{c-L}$$

leading to :

$$X_{1144} = [0.2373201571, 0.3224457580, 0.4402340851]$$

This point lies on the hyperbola $\{A, B, C, X(1), X(6)\} = CC\,(X_{649})$. Indeed, X(1144) lies on the open arc from X(1) to the vertex of ABC opposite the shortest side. (Jean-Pierre Ehrmann, 12/16/01)

23.2.5 X(2061)

Nothing special, but the length... 743 with best efforts. Point with only one relation. Shortened using Conway symbols.

23.3 orthopoint, points on circles

Remark 23.3.1. Orthopoint. Only 6 orthopoint pairs are registered, but 48 can be computed (involving 96 points on $\mathcal{L}_\infty$, that contains 229 named points). Has been abbreviated into **ho**rtopoint since "or" is a reserved word.

Remark 23.3.2. Inverse in a given circle. Points on $\mathcal{L}_\infty$ are not considered here.

1. circumcircle. There are 352 named points whose inverse in circumcircle is named too. Among them, 258 are on the circumcircle itself, and there are 47 pairs of "true" inverse pairs. Among these 94 points, 29 aren't listed.
   On the circumcircle, 220 named points have a named isogonal conjugate (among the 229 points of $\mathcal{L}_\infty$).
   62 antipodal pairs are listed.

2. incircle. There are 53 named points whose inverse in incircle is named too. Among them, 39 are on the incircle itself, and there are 7 pairs of "true" inverse pairs. Among these 14 points, 3 aren't listed.

3. nine points circle. There are 55 named points whose inverse in the nine points circle is named too. Among them, 37 are on the nine points circle itself, and there are 9 pairs of "true" inverse pairs. Among these 18 points, 4 aren't listed.

4. Bevan circle : 19, 9, 5 (pairs).

5. Brocard circle : 98, 4, 47 (pairs). Among 94 points, only 89 are listed.

6. Spieker circle : 8, 8, 0 (pairs)

7. Orthocentroidal circle : 82, 2, 40 (pairs). Among the 80 points, 20 aren't listed.

8. Fuhrmann circle : 12, 2, 5 (pairs). Among the 10 points, only 4 are listed (and belong to 4 different pairs).

9. First Lemoine : 16, 2, 7 (pairs).

10. Second Lemoine : 12, 2, 5 (pairs).

Remark 23.3.3. Antipode in circumcircle

1. From a total of 258 named points on the circle, 124 have a named antipode.

2. Antipode is mostly given as a member of the list "reflection", that appears as I_3 in the corresponding field. 10 are missing.

Remark 23.3.4. Antipode in nine points circle

1. Among 37 named points on the nine points circle, 24 have a named antipode (12 pairs).

2. Among 37 named points on γ, 33 have a named anticomplement

23.4 points on lines

Remark 23.4.1. Bad extractions. Due to lack of regularity (may be human modifications), 50 lists of lines were badly extracted (mostly : HTML tags, reflection, extra spaces).
`SELECT num, gerade FROM center WHERE gerade like "%\_ %"`
Lack of pattern for these :

505	680	1154	1180	1300	1573	1593	1689	1944	1951	2051
2173	2388	2442	2503	2708	2719	2723	2725	2727	2758	2776
2974	3037	3110	3111	3113	3308	3340	3354	3363	3501	

Remark 23.4.2. There are 32322 quotations, involving 15988 lines.

1. Lines are referred by two *other* points, so that none of the 169 lines quoted as going through X_1 are quoted under their "true" name.

2. Here are the statistics :

1	9436	9	43	17	3	25	2	43	1	184	1
2	2700	10	32	18	4	26	2	47	1	229	1
3	2207	11	24	19	5	27	1	62	1	312	1
4	725	12	19	20	2	29	2	75	1		
5	375	13	14	21	2	30	1	76	1		
6	151	14	7	22	5	35	2	78	1		
7	110	15	4	23	1	36	2	81	1		
8	60	16	10	24	3	38	1	100	1		

   This has to be read as : there are 9446 lines that are quoted only once, and so on.

3. In fact, most of the lines involving only three points are quoted only at their third point (except from 120 of them that are not quoted at the third point). For example, X_1 is involved in 294 quoted lines whereas only 169 of them are quoted in the X_1 page. When taking this fact into account, we obtain :

		9	44	17	3	25	2	43	1	184	1
		10	34	18	4	26	2	47	1	229	1
3	12998	11	25	19	5	27	1	62	1	312	1
4	1773	12	20	20	2	29	2	75	1		
5	580	13	14	21	2	30	1	76	1		
6	220	14	7	22	4	35	2	78	1		
7	119	15	4	23	2	36	2	81	1		
8	69	16	10	24	3	38	1	100	1		

4. There are 1195 lines that goes through at least five named points.

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