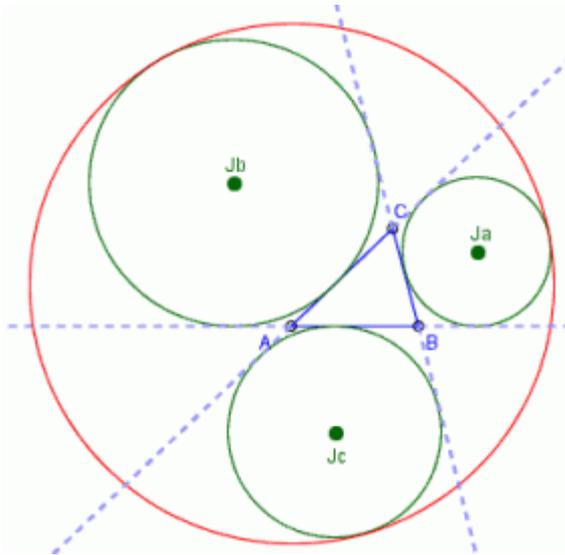


Apollonius Circle

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Abstract. We illustrate how the reader can use the results of the encyclopedia [1] and other computer-generated results. The computer-generated results, quoted in this article and not included in the first edition of [1], will be included in the next editions.

The Apollonius circle of a triangle is the circle tangent internally to each of the three excircles. See the figure:



There are many ways the Apollonius circle to be constructed by using straightedge and compass. The famous *Apollonius problem* for three circles states: Given three arbitrary circles, to construct the circles tangent to each of them. There are a few methods to solve the problem. See e.g. Dekov Software [Geometric Constructions](#). To construct the Apollonius circle we can use one of these methods.

There are a few additional ways to construct the Apollonius circle. These additional methods are based on the fact that the given circles are not arbitrary, but they are the excircles of a given triangle. We shall see a few such methods below.

The reader may consult Dekov Software [Geometric Constructions](#) for detailed description of constructions.

1. Construct the center and the radius

If we can construct the center and the radius of a circle, we can construct the circle. From

page Theorems, Points, Center of the Apollonius Circle, [1], we see that we can construct the center of the Apollonius circle as the intersection point of the Apollonius line and the Brocard axis (known result, see Kimberling's ETC [2]). It known that the radius of the Apollonius circle is equal to

$$(r + s.s/r)/4$$

(see e.g. M. R. Stevanovic [3]) We can construct the radius. Hence, we can construct the Apollonius circle.

Note that in the methods below we have not to construct the excircles.

2. Construct the center and a point on the circle

We can construct the center of the Apollonius circle (see the previous section). Then we can construct one excircle, e.g., the A-Excircle. The intersection point of the line passing through the center of the Apollonius circle and the A-Excircle lies on the Apollonius circle (computer generated result; this result is obvious). Hence, we can construct the Apollonius circle.

3. Construct three points of the circle

If we can construct three points of a circle, then we can construct the circle as the circle passing through these three points. The three tangency points of the Apollonius circle and the excircles are the vertices of the Apollonius triangle. Hence, we can try to construct the Apollonius triangle, and then to construct its circumcircle, that is, the Apollonius circle.

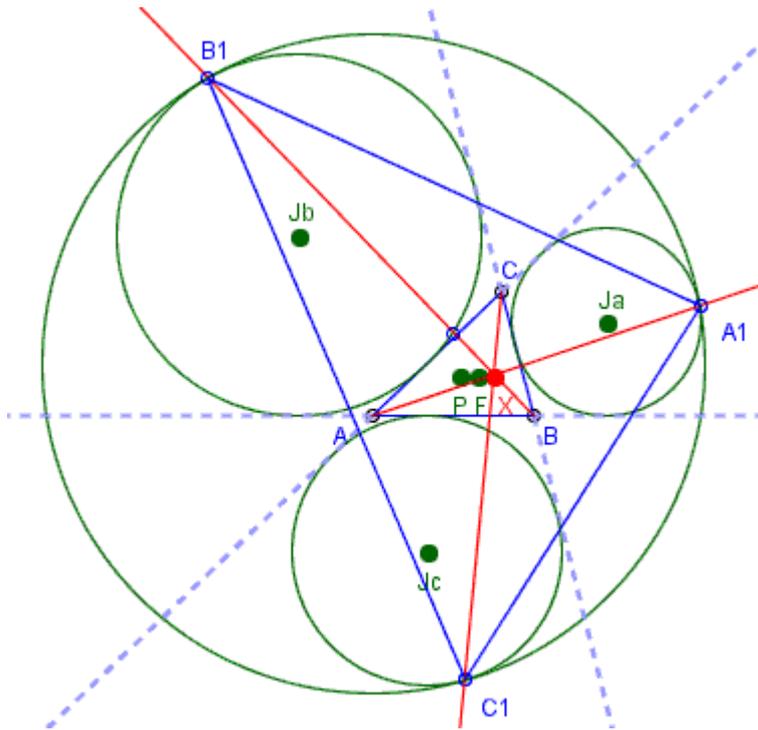
There are many methods to construct a triangle. We can try to use the following method: If a triangle is perspective with two other triangles and if we can construct these triangles and the respective perspector, then we can construct the triangle. The Apollonius triangle is perspective with

- (1) triangle ABC; the perspector is the Apollonius point (this is the definition of the Apollonius point);
- (2) Feuerbach triangle; the perspector is the Spieker center (computer-generated result, not included in the first edition of [1]; is this result known ?)
- (3) Symmedial triangle; the perspector is the isotomic complement of the orthocenter of the intouch triangle (computer-generated result, not included in the first edition of [1]; is this result known ?)
- (4) Excentral triangle; the perspector is the center of the Apollonius circle (computer-generated result, not included in the first edition of [1]; this result is obvious).

From page Theorems, Points, Apollonius Point, we can see a few ways to construct of the Apollonius point:

- (1) Construct the Apollonius point as the isogonal conjugate of the isotomic conjugate of the

Second Feuerbach point (computer-generated result; is this result known ?) See the figure:



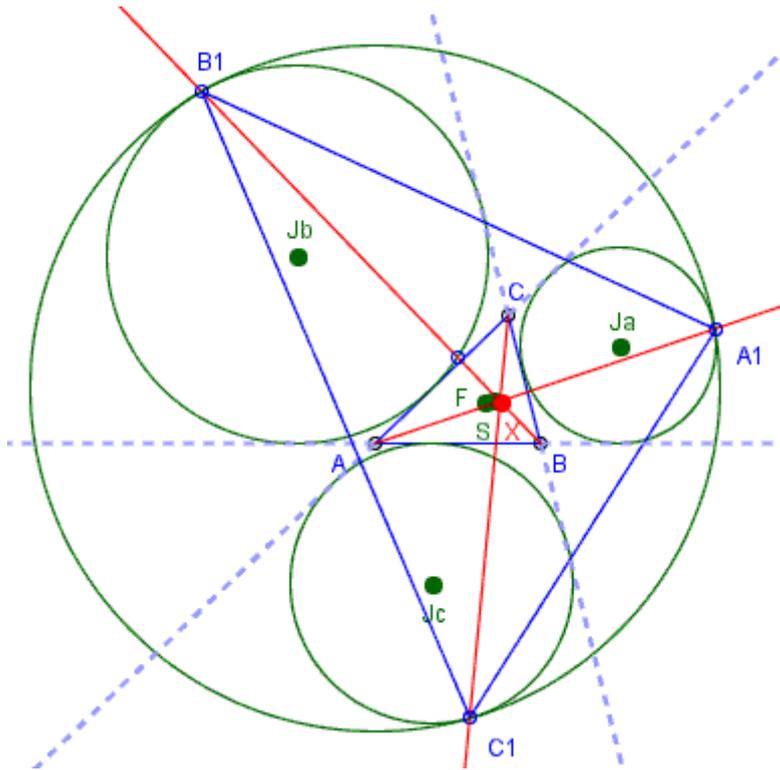
F - Second Feuerbach point.

P - Isotomic conjugate of F.

X - Isogonal conjugate of P = Apollonius point.

$\triangle A_1B_1C_1$ - Apollonius triangle.

(2) Construct the Apollonius Point as the Yiu conjugate of the symmedian point and the Second Feuerbach point (computer-generated result; is this result known ?) See the figure:

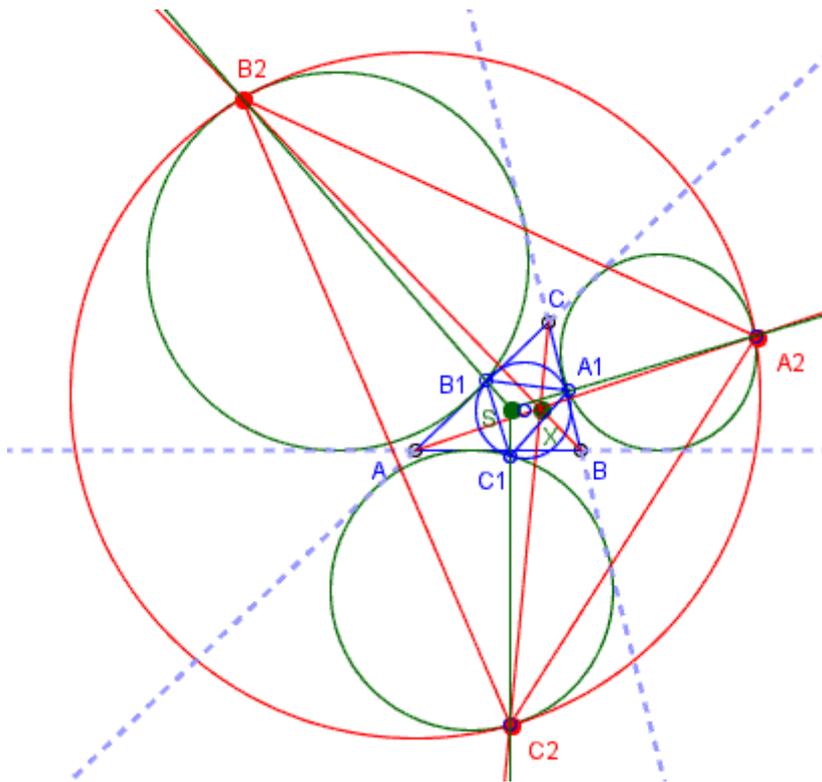


- F - Second Feuerbach point.
- S - Symmedian point.
- X - Yiu conjugate of F and S = Apollonius point.
- $\triangle A_1B_1C_1$ - Apollonius triangle.

(3) Construct the Apollonius point as the point of intersection of the line passing through the incenter and the center of the Apollonius circle (the Apollonius line), and the line passing through the circumcenter and the symmedian point (the Brocard axis) (computer-generated result, the result is known, see C.Kimberling [2]).

We can construct the Apollonius triangle by using any pair of triangles listed above. The four triangles give us 6 ways to construct the Apollonius triangle. (An extended computer research would give us probably a few additional triangles.)

If we take the first two triangles in the above list, that is, triangle ABC and the Feuerbach triangle $\triangle A_1B_1C_1$, we can construct the Apollonius triangle $\triangle A_2B_2C_2$ as follows: Construct the Apollonius point X and the Spieker center S. Construct A_2 is the point of intersection of lines AX and SA_1 . Similarly, construct B_2 and C_2 . Now we can construct the Apollonius circle as the circumcircle of the Apollonius triangle. See the figure:



- S - Spieker center.
- X - Apollonius point.
- $\triangle A_1B_1C_1$ - Feuerbach triangle.
- $\triangle A_2B_2C_2$ - Apollonius triangle.

4. Apollonius circle as the inverse image of a circle

A theorem from page Theorems, Circles, Apollonius Circle, [1], states that the Apollonius circle is the inverse of the Nine-point circle with respect to the radical circle of the excircles. Hence, we can construct the Apollonius circle.

The above result is known (P. Yiu [4]), but the next result probably is not known. If we ask a computer to make a deeper investigation, we will receive the following result (not included in the first edition of [1]; is this result known?):

The Apollonius circle is congruent to the inverse circle of the Bevan circle with respect to the radical circle of the excircles of the anticomplementary triangle.

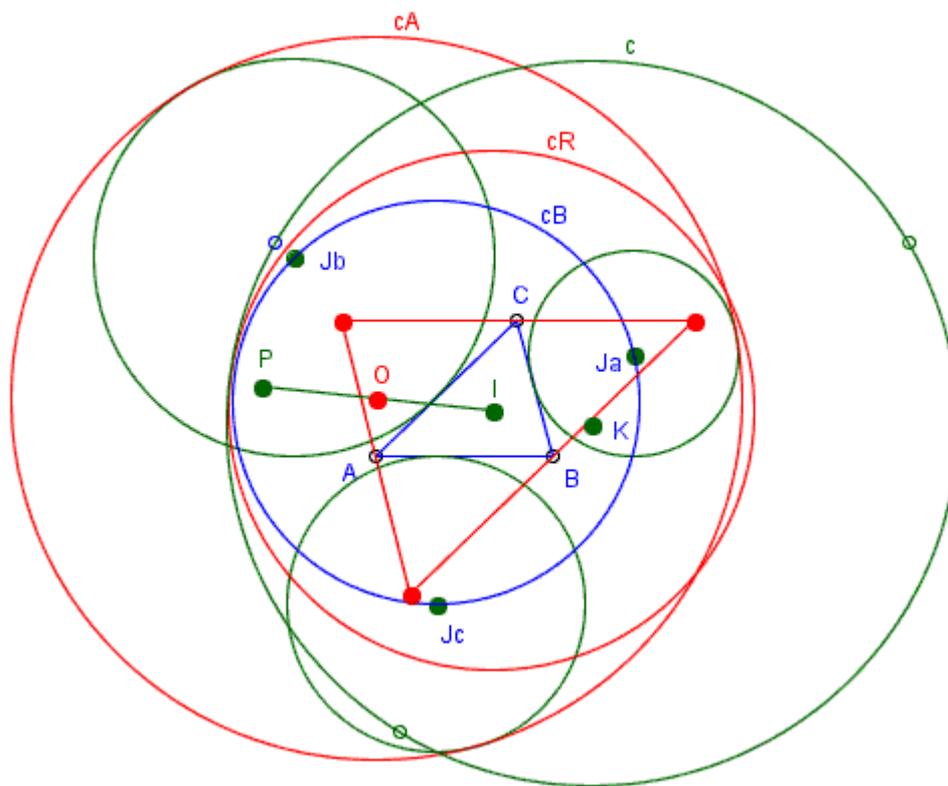
Label by c the inverse circle of the Bevan circle with respect to the radical circle of the excircles of the anticomplementary triangle.

Now we need the relationship between two points: the center of the Apollonius circle and the center of circle c . We ask again the computer and receive a few relationships, e.g.,

The center of circle c is the complement of the reflection of the incenter in the center of the Apollonius circle (this result is not included in the first edition of [1]; is this result known?).

Now we can construct the Apollonius circle as follows. First, construct circle c . Then

construct the center of the Apollonius circle as the midpoint of the incenter and the anticomplement of the center of circle c . Finally, construct the Apollonius circle. See the figure:



J_a, J_b, J_c - centers of the excircles.

cB - Bevan circle.

The red triangle - Anticomplementary triangle.

cR - Radical circle of the excircles of the anticomplementary triangle.

c - inverse circle of the Bevan circle with respect to the radical circle of the excircles of the anticomplementary triangle.

The three points on circle c are the inverse images of J_a, J_b, J_c with respect to circle cR .

(First we construct these three points, then we construct circle c as the circle passing through these points.)

K - center of circle c .

P - anticomplement of K .

I - Incenter.

O - midpoint of segment IP = center of the Apollonius circle.

cA - Apollonius circle = circle with center O and radius equal to the radius of circle c .

5. The use of the similitude centers of the Apollonius circle and another circle

If we can construct a circle and the two similitude centers of the circle and the Apollonius circle, we can construct the Apollonius circle as follows. First construct the center of the Apollonius circle as the harmonic conjugate of the center of the constructed circle with respect to the similitude centers. Then construct the Apollonius circle.

For example, we can use the circumcircle. The similitude centers could be constructed as follows: Construct the internal similitude center of the circumcircle and the Apollonius circle as the intersection point of the line passing through the circumcenter and the symmedian point (the Brocard axis), and the line passing through the orthocenter and the mittenpunkt. (computer-generated result, not included in the first edition of [1]; the result is known, see C.Kimberling [2]). Construct the external similitude center of the circumcircle and the Apollonius circle as the intersection point of the line passing through the incenter and the centroid (the Nagel line), and the line passing through the circumcenter and the symmedian point (the Brocard axis). (computer-generated result, not included in the first edition of [1]; the result is known, see C.Kimberling [2]). Now construct the center of the Apollonius circle as the harmonic conjugate of the circumcenter with respect to the similitude centers, and then construct the Apollonius circle.

We can use a number of other circles in the place of the circumcircle.

Concluding Remarks

The methods above could be summarized to the following general method. If we have to construct a geometric object, first we have to ask computer about the properties of this object. A computer program can answer the question. If we need some additional information, we can ask again, and so on. Then we can use the properties to construct the object. All above constructions could be obtained by this way. Hence, we can replace the methods used by now (inventiveness) with a new method - questions and answers. In fact, the computer will solve the problem for us. Hence, instead of inventiveness, now we have an ALGORITHM for finding a solution of the construction problem.

Thanks

The figures in this article are produced by using the program C.a.R. (Compass and Ruler), an amazing program created by Rene Grothmann. The Grothmann's program is available for download in the Web: [Rene Grothmann's C.a.R.](http://www.grothmann.com/). It is free and open source. The reader may verify easily the theorems of the encyclopedia [1] by using C.a.R. Many thanks to Rene Grothmann for his wonderful program.

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