Apollonius Problem

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Abstract. Given three non-intersecting circles, there are eight circles that are tangent to each of the given circles. We note that these eight circles form four pairs of circles such that the circles of a pair are inverses in the radical circle of the given three circles.

The famous Apollonius problem includes the following problem: Given three non-intersecting circles, to construct all circles that are tangent to each of the given circles. There are eight total solutions. Construction of these eight circles, using the method due to Joseph Gergonne (1771-1859), is given, e.g. in Geometric Constructions [5].

Denote by 1 and 2 the Apollonius circles tangent respective externally and internally to the three given circles.

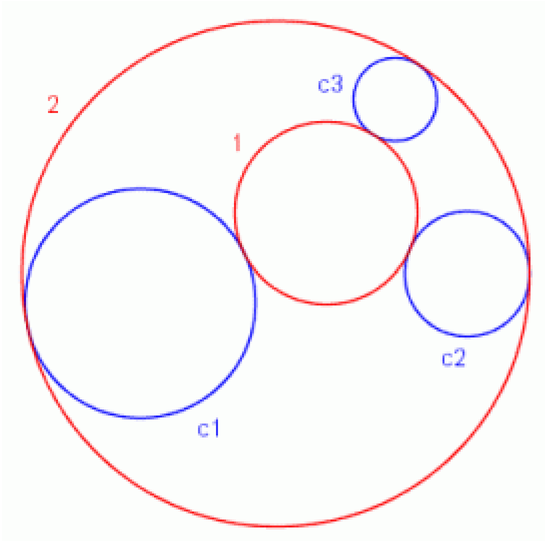
If the given three circles are the three excircles of a given triangle, then circle 1 is the nine-point circle and circle 2 is the Apollonius circle of the given triangle. Paul Yiu [9] (see also Grinberg and Yiu [3]) noticed that the Apollonius circle of the given triangle is inverse image of the nine-point circle in the radical circle of the excircles.

If the given three circles are the three Lucas circles of a given triangle, then circle 1 is the inner Soddy circle of the Lucas circles and circle 2 is the outer Soddy circle of the Lucas circles. In this case circle 2 coincides with the circumcircle of the given triangle. Peter Moses [6] noticed that the inner Soddy circle of the Lucas circles is the inverse image of the circumcircle of the given triangle.

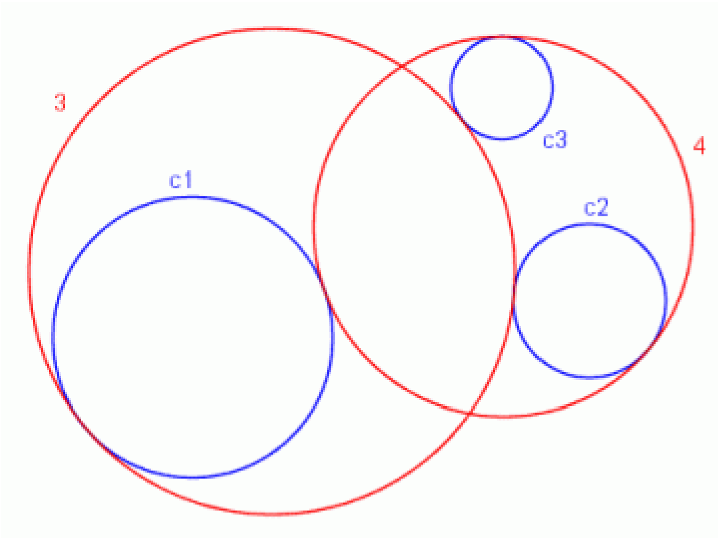
The computer research of the Apollonius problem gave the following result: Given three arbitrary non-intersecting circles. Then Apollonius circle 1 is the inverse image of Apollonius circle 2 in the radical circle of the given three circles. Hence, the above results are valid in the general case of arbitrary circles. This result seems to be obvious. I note it here since in the literature available for me ([1],[2],[3],[6],[7],[8],[9]), I could not see it.

Label the given circles c1, c2 and c3. Label solutions as follows (We follow notations from Geometric Constructions [5]).

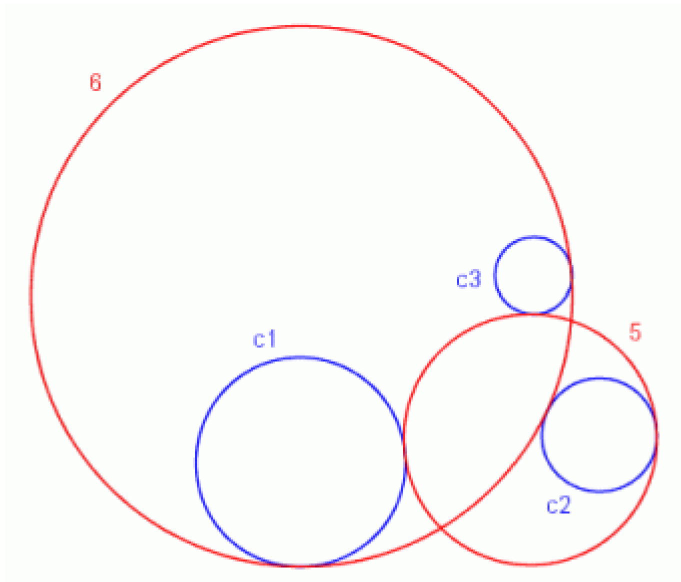
Circle 1 is tangent externally to c1, c2 and c3.
Circle 2 is tangent internally to c1, c2 and c3.
See the figure:



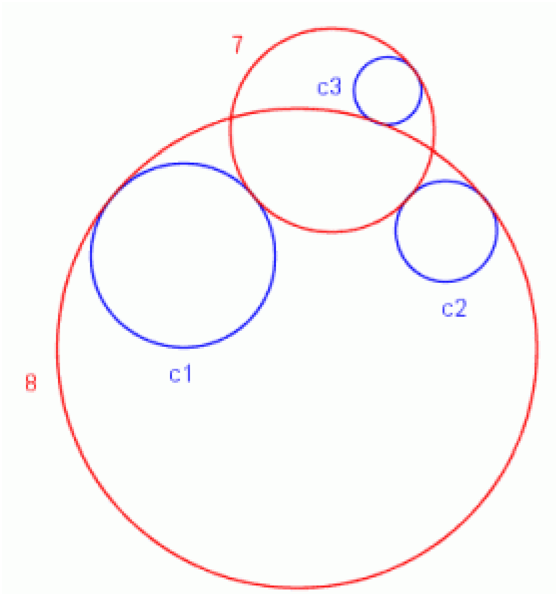
Circle 3 is tangent internally to c1 and externally to c2 and c3.
Circle 4 is tangent externally to c1 and internally to c2 and c3.
See the figure:



Circle 5 is tangent internally to c2 and externally to c1 and c3.
Circle 6 is tangent externally to c2 and internally to c1 and c3.
See the figure:



Circle 7 is tangent internally to c3 and externally to c1 and c2.
Circle 8 is tangent externally to c3 and internally to c1 and c2.
See the figure:



Let the radical circle of the given three circles serves as an inversion circle.Then

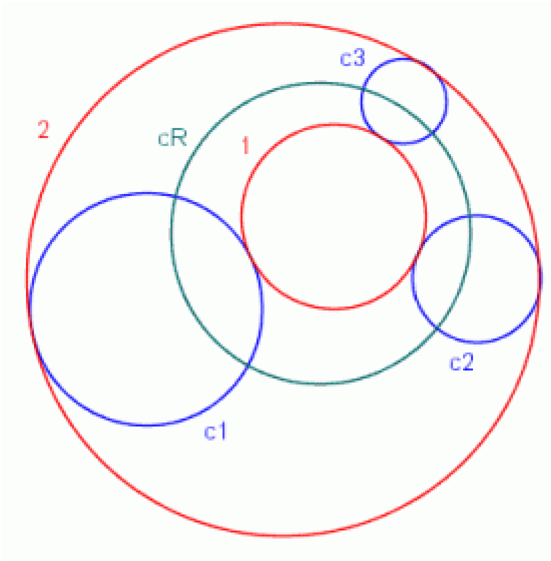
Circle 1 is the inverse image of circle 2.
Circle 3 is the inverse image of circle 4.
Circle 5 is the inverse image of circle 6.
Circle 7 is the inverse image of circle 8.

In all these cases, a point of tangency transforms to a point of tangency. E.g., the point of tangency of circles c1 and 1 is the inverse image of the point of tangency of circles c1 and 2.

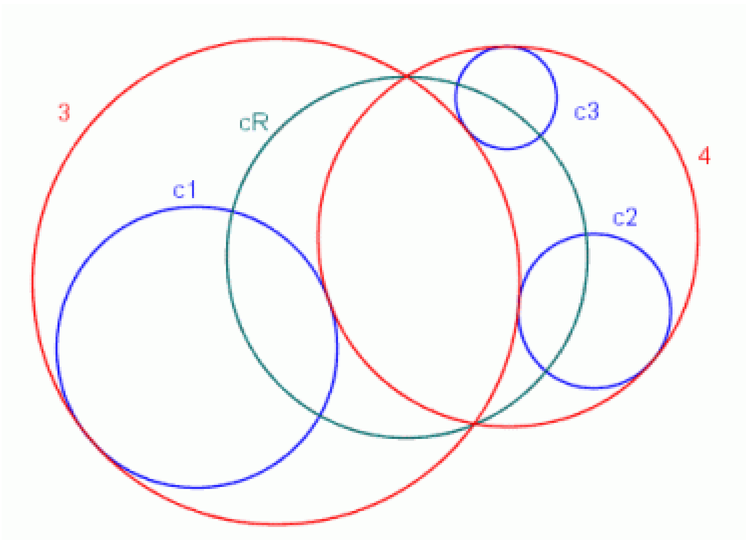
Recall that if a circle A is the inverse image of a circle B in a circle C, then circle B is the inverse image of circle A in circle C, too.

See the figures (cR is the radical circle of the given three circles c1, c2 and c3):

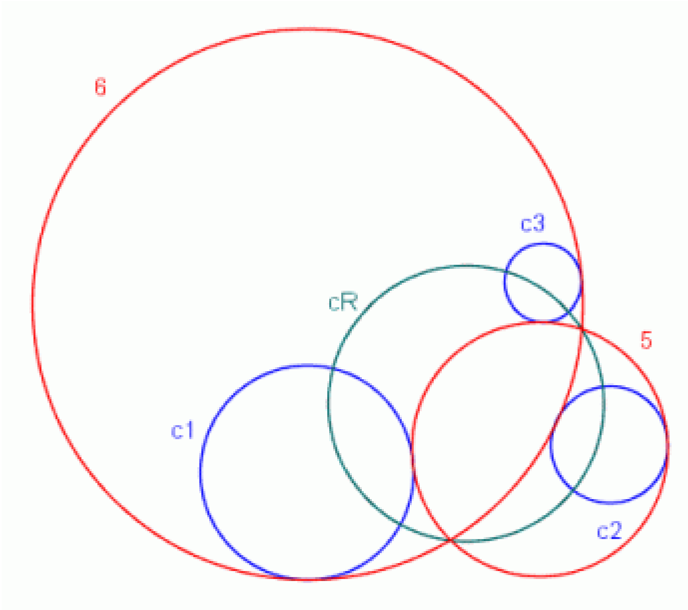
Circle 1 is the inverse image of circle 2 in the radical circle of circles c1, c2 and c3:



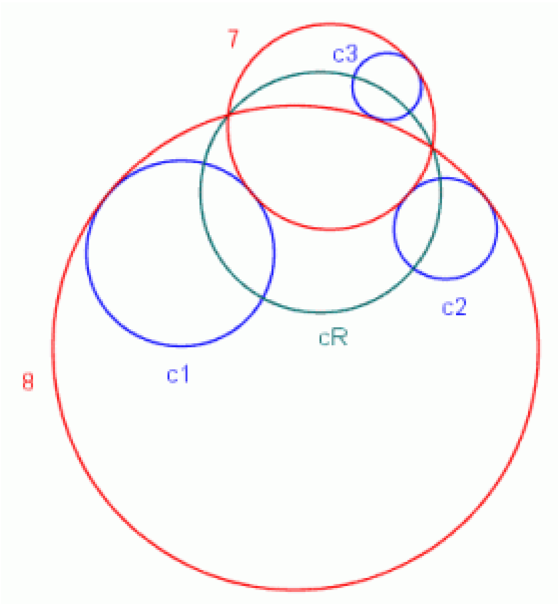
Circle 3 is the inverse image of circle 4 in the radical circle of circles c1, c2 and c3:



Circle 5 is the inverse image of circle 6 in the radical circle of circles c1, c2 and c3:

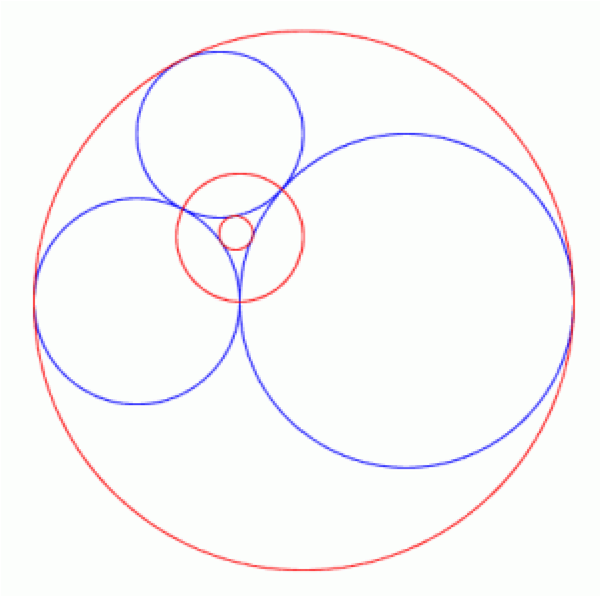


Circle 7 is the inverse image of circle 8 in the radical circle of circles c1, c2 and c3:

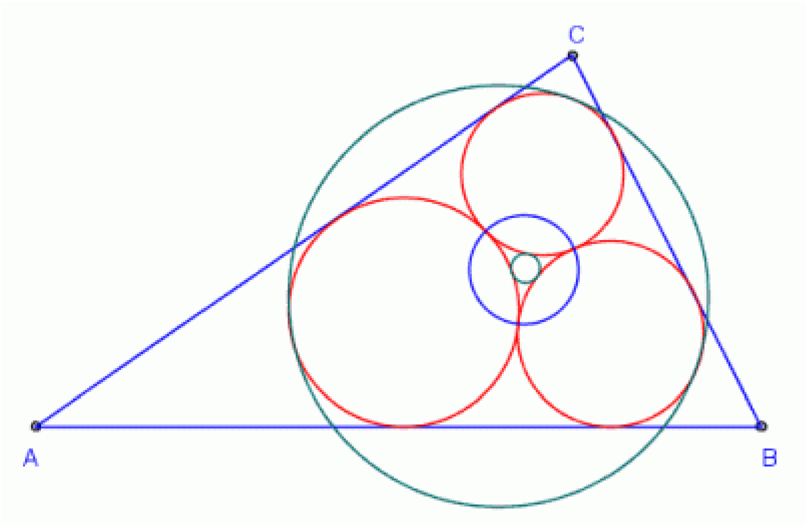


There are a few special cases of the Apollonius problem.

If the three given circles are tangent each other pairwise, the Apollonius problem has only two solutions - circles 1 and 2. In this case, circle 1 is knows as the inner Soddy circle and circle 2 is known as the outer Soddy circle. Construction of these two circles, using the method due to David Eppstein [1], is given, e.g. in Geometric Constructions [5]. It is enough we to construct one of the Soddy circles, then the other circle can be constructed as inverse image. See the figure:



We note the special case of the inner and outer Soddy circles of Malfatti circles. See the figure:



Thanks

The figures in this article are produced by using the program C.a.R. (Compass and Ruler), an amazing program created by Rene Grothmann. The Grothmann's program is available for download in the Web: Rene Grothmann's C.a.R.. It is free and open source. The reader may verify easily the theorems of the encyclopedia [4] and the statements of this article by using C.a.R. Many thanks to Rene Grothmann for his wonderful program.

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