Journal of Computer-Generated Euclidean Geometry

Computer-Generated Mathematics: The Gergonne Point

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Abstract. We illustrate the use of the computer program "Machine for Questions and Answers" (The Machine) for discovering of new theorems in Euclidean Geometry. The paper contains more than 20 new theorems about the Gergonne Point, discovered by the Machine.

Keywords: computer-generated mathematics, Euclidean geometry

"Within ten years a digital computer will discover and prove an important mathematical theorem." (Simon and Newell, 1958).

This is the famous prediction by Simon and Newell [1]. Now is 2008, 50 years later. The first computer program able easily to discover new deep mathematical theorems - The *Machine for Questions and Answers* (The *Machine*) [2,3] has been created by the author of this article, in 2006, that is, 48 years after the prediction. The Machine has discovered a few thousands new mathematical theorems, that is, more than 90% of the new mathematical computer-generated theorems since the prediction by Simon and Newell. In 2006, the Machine has produced the first computer-generated encyclopedia [2].

Given an object (point, triangle, circle, line, etc.), the Machine produces theorems related to the properties of the object. The theorems produced by the Machine are either known theorems, or possible new theorems. A *possible new* theorem means that the theorem is either known theorem, but the source is not available for the author of the Machine, or the theorem is a new theorem.

The advantages in using the Machine are as follows. (1) It is not necessary we to be inventive and even it is not necessary we to think. It is enough we to click with the mouse in order to obtain the theorems. (2) The Machine produces complete knowledge. If there exists a theorem related to the object, the Machine discovers the theorem. (3) The people make errors, but the computers do not make errors.

In this paper we illustrate the use of the Machine. We present two lists with possible new theorems about the Gergonne point, discovered by the Machine. These lists contain 57 possible new theorems. I expect that approximately 20 to 30 of these theorems are new theorems. Hence, the paper contains at least than 20 new theorems about the Gergonne Point, discovered by the Machine. The reader is invited to select the new theorems and to prove them. There are a few additional lists of theorems about the Gergonne point, produced

by the Machine, which are not included in this paper.

The Machine is at a very early stage of development. The current version of the Machine could be considered as a first step to an improved and extended version, able to discover new theorems and to produce encyclopedias in all branches of mathematics.

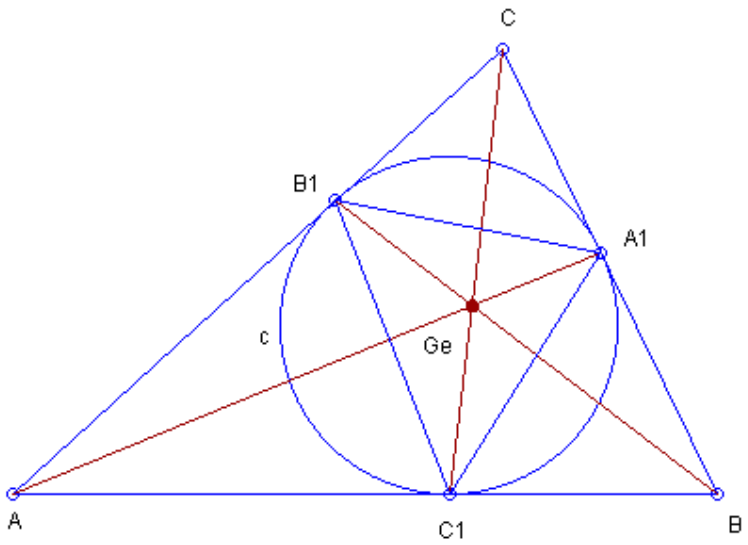
The reader may find the definitions used in this paper, in [2-4].

Intelligent Pedagogy

The Machine could be useful for students and teachers mainly in these directions: (1) The Machine could produce an encyclopedia of Euclidean geometry suitable for school students and teachers. (2) The Machine will give to the school students and teachers the possibility to discover new theorems. (3) The interactive use of the Machine will give to the school students and teachers the possibility to investigate in depth selected problems. (4) The Machine will give to the teachers the possibility easily to produce problems and theorems for textbooks, for use in the classroom, for home works, etc. (5) The Machine will give to the school students and teachers the possibility better to understand the abilities of computers to discover new theorems. Today's school students will produce the next computer-generated encyclopedias.

The Gergonne point

Recall the definition of the Gergonne Point. Given a triangle ABC, denote by A_1 the tangency point of the Incircle and side BC, by B_1 the tangency point of the Incircle and side CA, and by C_1 the tangency point of the Incircle and side AB. Lines AA_1, BB_1 and CC_1 concur in a point, called the *Gergonne Point*. Triangle $A_1B_1C_1$ is called the *Intouch Triangle*. In another terminology, the Intouch Triangle is the Cevian Triangle of the Gergonne Point, or equivalently, the Gergonne Point is the perspector of triangle ABC and the Intouch Triangle. See the Figure:



Roles of the Gergonne Point

The list below contains a few theorems about the Roles of the Gergonne Point, discovered by the Machine.

1. The Gergonne Point is the Mittenpunkt of the Orthic Triangle of the Intouch Triangle.
2. The Gergonne Point is the Symmedian Point of the Excentral Triangle of the Anticomplementary Triangle.
3. The Gergonne Point is the Symmedian Point of the Circum-Orthic Triangle of the Mid-Arc Triangle.
4. The Gergonne Point is the Symmedian Point of the Fourth Brocard Triangle of the Intouch Triangle.
5. The Gergonne Point is the Steiner Point of the First Brocard Triangle of the Intouch Triangle.
6. The Gergonne Point is the Center of the Brocard Circle of the Triangle of the reflections of the Incenter in the sides of Triangle ABC.
7. The Gergonne Point is the Circumcenter of the Triangle of the reflections of the Internal Center of Similitude of the Incircle and the Circumcircle in the sides of Triangle ABC.
8. The Gergonne Point is the Perspector of the Anticevian Triangle of the Gergonne Point and the Pedal Triangle of the Incenter.
9. The Gergonne Point is the Perspector of the Pedal Triangle of the Incenter and the Circumcevian Triangle of the Gergonne Point.
10. The Gergonne Point is the Perspector of Triangle ABC and the Excentral Triangle of the Anticomplementary Triangle.
11. The Gergonne Point is the Perspector of Triangle ABC and the Second Brocard Triangle of the Intouch Triangle.
12. The Gergonne Point is the Perspector of the Intouch Triangle and the Symmedial Triangle of the Intouch Triangle.
13. The Gergonne Point is the Homothetic Center of the Intouch Triangle and the Excentral Triangle of the Anticomplementary Triangle.
14. The Gergonne Point is the Perspector of the Intouch Triangle and the Second Brocard Triangle of the Intouch Triangle.
15. The Gergonne Point is the Homothetic Center of the Anticomplementary Triangle and the Orthic Triangle of the Intouch Triangle.
16. The Gergonne Point is the Homothetic Center of Triangle ABC and the Medial Triangle of the Orthic Triangle of the Intouch Triangle.
17. The Gergonne Point is the Perspector of Triangle ABC and the Symmedial Triangle of the Excentral Triangle of the Anticomplementary Triangle.
18. The Gergonne Point is the Homothetic Center of Triangle ABC and the Tangential Triangle of the Excentral Triangle of the Anticomplementary Triangle.
19. The Gergonne Point is the Perspector of the Intouch Triangle and the Symmedial Triangle of the Excentral Triangle of the Anticomplementary Triangle.
20. The Gergonne Point is the Perspector of the Intouch Triangle and the Tangential Triangle of the Excentral Triangle of the Anticomplementary Triangle.
21. The Gergonne Point is the Perspector of the Excentral Triangle and the Medial Triangle of the Symmedial Triangle of the Intouch Triangle.
22. The Gergonne Point is the Perspector of Triangle ABC and the Stevanovic Triangle of the Gergonne Points of the Triangulation triangles of the Center of the Outer

Soddy Circle.

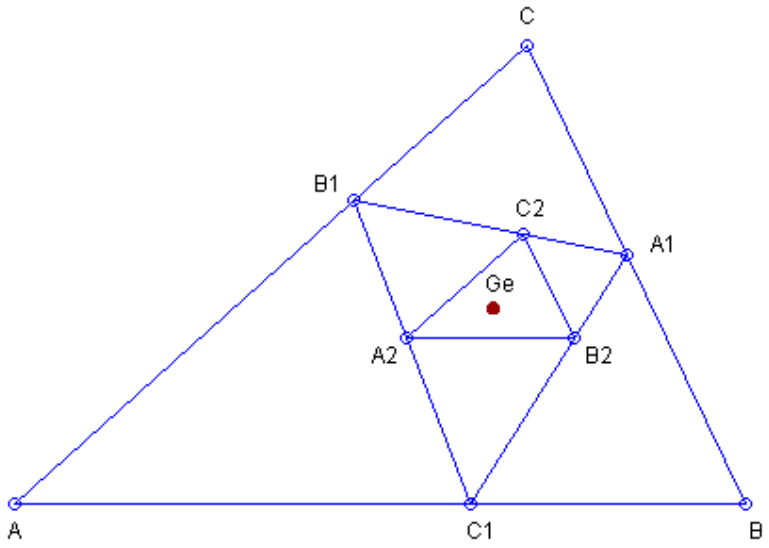
23. The Gergonne Point is the Perspector of Triangle ABC and the Stevanovic Triangle of the Gergonne Points of the Triangulation triangles of the Center of the Inner Soddy Circle.
24. The Gergonne Point is the Homothetic Center of Triangle ABC and the Triangle of the Gergonne Points of the Corner Triangles of the Centroid.
25. The Gergonne Point is the Perspector of Triangle ABC and the Triangle of the Internal Centers of Similitude of the Incircles and the Circumcircles of the Corner Triangles of the Orthocenter.
26. The Gergonne Point is the Perspector of Triangle ABC and the Triangle of the reflections of the Internal Center of Similitude of the Incircle and the Circumcircle in the sides of the Excentral Triangle.
27. The Gergonne Point is the Perspector of Triangle ABC and the Triangle of the reflections of the vertices of the Extouch Triangle in the Mittenpunkt.
28. The Gergonne Point is the Homothetic Center of Triangle ABC and the Triangle of the reflections of the vertices of the Anticomplementary Triangle in the Mittenpunkt.
29. The Gergonne Point is the Homothetic Center of the Medial Triangle and the Triangle of the Centroids of the Triangulation Triangles of the Gergonne Point.
30. The Gergonne Point is the Homothetic Center of the Medial Triangle and the Triangle of the Gergonne Points of the Anticevian Corner Triangles of the Centroid.
31. The Gergonne Point is the Perspector of the Intouch Triangle and the Triangle of the Gergonne Points of the Corner Triangles of the Centroid.
32. The Gergonne Point is the Perspector of the Intouch Triangle and the Triangle of the Internal Centers of Similitude of the Incircles and the Circumcircles of the Corner Triangles of the Orthocenter.
33. The Gergonne Point is the Perspector of the Intouch Triangle and the Triangle of the reflections of the Internal Center of Similitude of the Incircle and the Circumcircle in the sides of the Excentral Triangle.
34. The Gergonne Point is the Homothetic Center of the Anticomplementary Triangle and the Triangle of the Mittenpunkts of the Anticevian Corner Triangles of the Centroid.
35. The Gergonne Point is the Homothetic Center of the Anticomplementary Triangle and the Triangle of the reflections of the Mittenpunkt in the vertices of the Medial Triangle.
36. The Gergonne Point is the Product of the Incenter and the Isotomic Conjugate of the Mittenpunkt.
37. The Gergonne Point is the Product of the External Center of Similitude of the Incircle and the Circumcircle and the Isotomic Conjugate of the Symmedian Point.
38. The Gergonne Point is the Product of the Isogonal Conjugate of the Mittenpunkt and the Isotomic Conjugate of the Incenter.
39. The Gergonne Point is the External Center of Similitude of the Outer Soddy Circle and the Radical Circle of the Lucas Circles of the Intouch Triangle.
40. The Gergonne Point is the Internal Center of Similitude of the Anticomplementary Circle and the Nine-Point Circle of the Intouch Triangle.
41. The Gergonne Point is the External Center of Similitude of the Adams Circle and the Lemoine Circle of the Intouch Triangle.
42. The Gergonne Point is the Center of the Cosine Circle of the Intouch Triangle.
43. The Gergonne Point lies on the Brocard Circle of the Intouch Triangle.

We illustrate a few of the above theorems. We invite the reader to select the new theorems

and to prove them.

Theorem 1. The Gergonne Point is the Mittenpunkt of the Orthic Triangle of the Intouch Triangle.

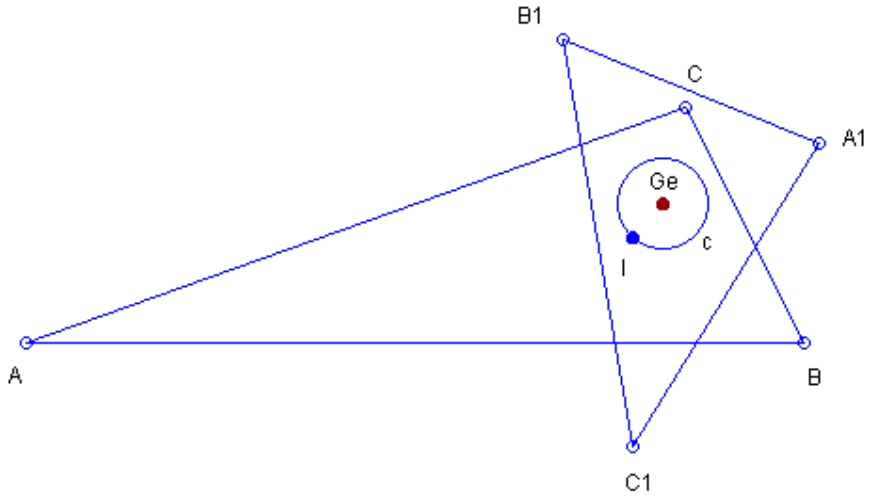
See the Figure:



Ge - Gergonne Point;
$A_1B_1C_1$ - Intouch Triangle;
$A_2B_2C_2$ - Orthic Triangle of the Intouch Triangle;
The Gergonne Point Ge is the Mittenpunkt of the Orthic Triangle of the Intouch Triangle.

Theorem 6. The Gergonne Point is the Center of the Brocard Circle of the Triangle of reflections of the Incenter in the sides of Triangle ABC.

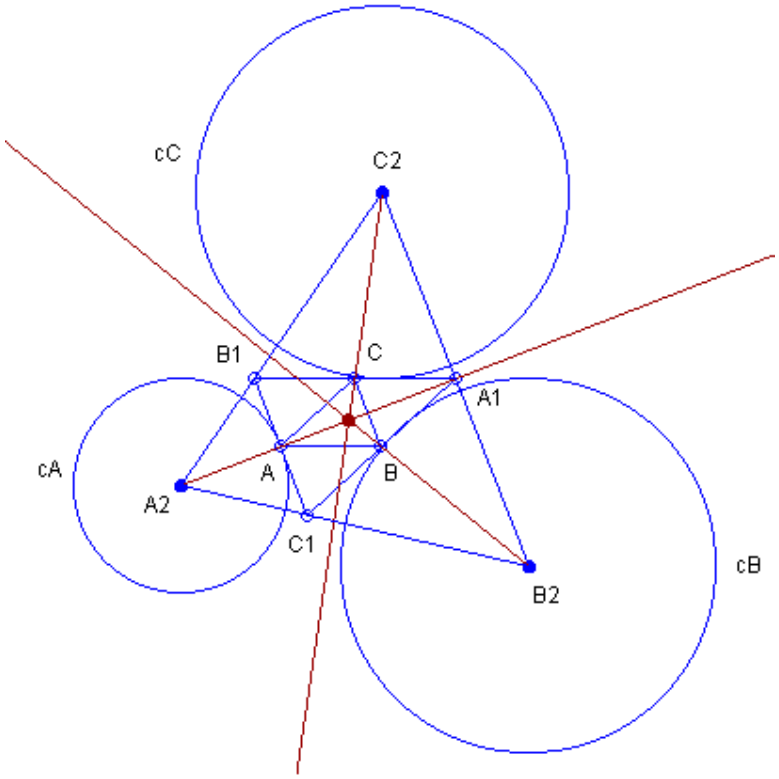
See the Figure:



Ge - Gergonne Point;
I - Incenter;
A_1 - reflection of point I in the side BC;
B_1 - reflection of point I in the side CA;
C_1 - reflection of point I in the side AB;
$A_1B_1C_1$ - Triangle of reflections of the Incenter in the sides of Triangle ABC;
c - Brocard Circle of triangle $A_1B_1C_1$;
The Gergonne Point Ge is the Center of the Brocard Circle of the Triangle of reflections of the Incenter in the sides of Triangle ABC.

Theorem 10. The Gergonne Point is the Perspector of Triangle ABC and the Excentral Triangle of the Anticomplementary Triangle.

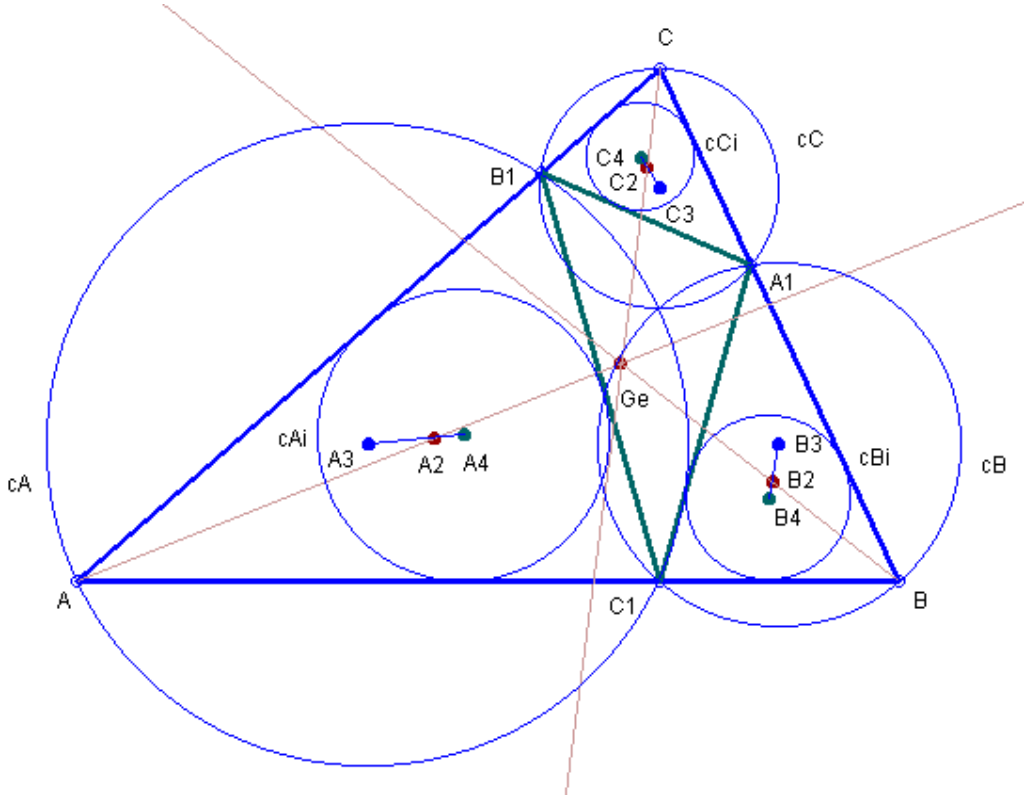
See the Figure:



The red point - Gergonne Point;
$A_1B_1C_1$ - Anticomplementary Triangle;
cA, cB, cC - Excircles of the Anticomplementary Triangle;
A_2, B_2, C_2 - Excenters of the Anticomplementary Triangle;
$A_2B_2C_2$ - Excentral Triangle of the Anticomplementary Triangle;
Lines AA_2, BB_2 and CC_2 concur in the Gergonne point, that is, the Gergonne Point is the Perspector of Triangle ABC and the Excentral Triangle of the Anticomplementary Triangle.

Theorem 25. The Gergonne Point is the Perspector of Triangle ABC and the Triangle of the Internal Centers of Similitude of the Incircles and the Circumcircles of the Corner Triangles of the Orthocenter.

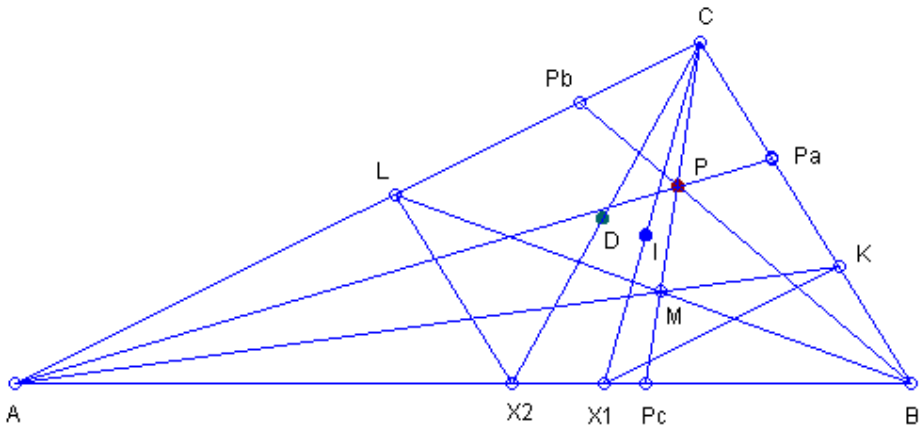
See the Figure:



Ge - Gergonne Point;
$A_1B_1C_1$ - Orthic Triangle = Cevian Triangles of the Orthocenter;
cA and A3 - Circumcircle and Circumcenter of triangle AB_1C_1;
cAi and A4 - Incircle and Incenter of triangle AB_1C_1;
A2 - Internal Centers of Similitude of circles cA and cAi;
cB and B3 - Circumcircle and Circumcenter of triangle BC_1A_1;
cBi and B4 - Incircle and Incenter of triangle BC_1A_1;
B2 - Internal Centers of Similitude of circles cB and cBi;
cC and C3 - Circumcircle and Circumcenter of triangle CA_1B_1;
cCi and C4 - Incircle and Incenter of triangle CA_1B_1;
C2 - Internal Centers of Similitude of circles cC and cCi;
$A_2B_2C_2$ - Triangle of the Internal Centers of Similitude of the Incircles and the Circumcircles of the Corner Triangles of the Orthocenter;
Lines AA_2, BB_2 and CC_2 concur in the Gergonne point, that is, the Gergonne Point is the Perspector of Triangle ABC and the Triangle of the Internal Centers of Similitude of the Incircles and the Circumcircles of the Corner Triangles of the Orthocenter.

Theorem 36. The Gergonne Point is the Product of the Incenter and the Isotomic Conjugate of the Mittenpunkt.

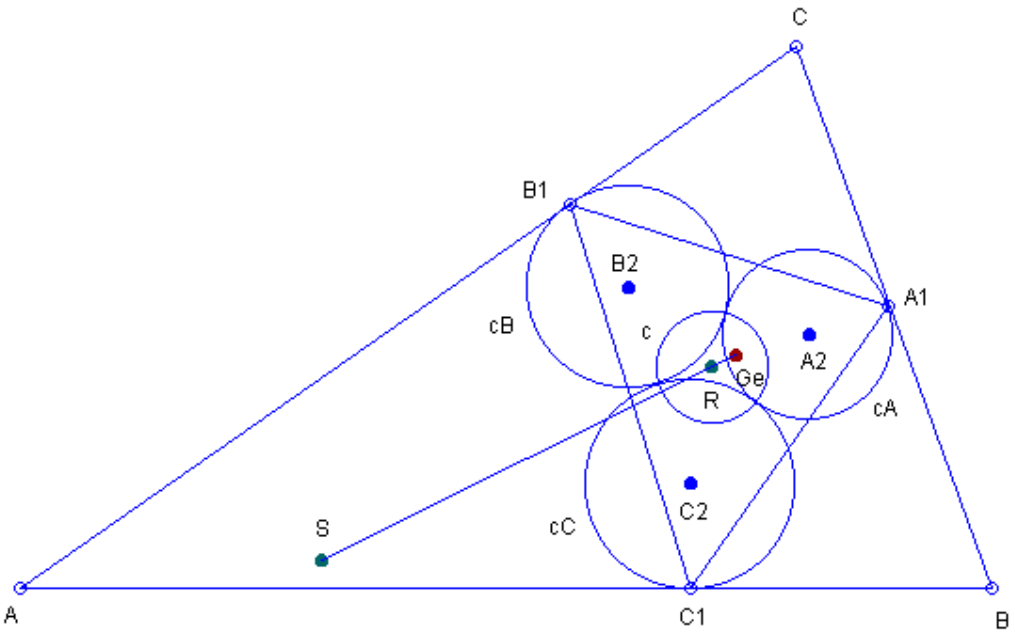
See the Figure:



I - Incenter;
D - Isotomic Conjugate of the Mittenpunkt;
P - Gergonne Point;
X1 - intersection point of lines AB and CI;
X2 - intersection point of lines AB and CD;
K - intersection point of line BC and the line through X1 parallel to line AC;
L - intersection point of line AC and the line through X2 parallel to line BC;
M - intersection point of lines AK and BL;
Pc - intersection point of lines AB and CM;
Similarly, construct points Pa and Pb;
P is the intersection point of lines APa, BPb and CPc, that is, the Gergonne Point is the Product of the Incenter and the Isotomic Conjugate of the Mittenpunkt

Theorem 39. The Gergonne Point is the External Center of Similitude of the Outer Soddy Circle and the Radical Circle of the Lucas Circles of the Intouch Triangle.

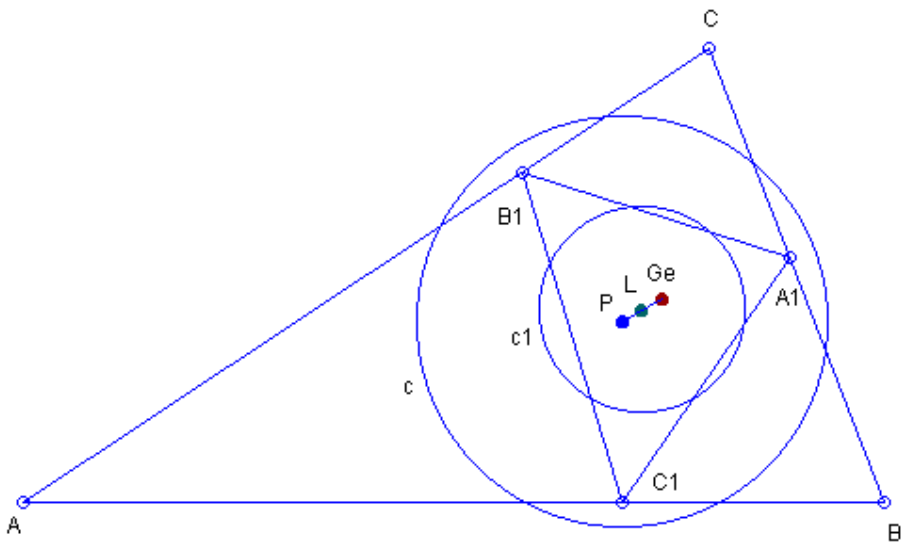
See the Figure:



Ge - Gergonne Point;
S - Center of the Outer Soddy Circle (the Outer Soddy Circle is outside the picture);
$A_1B_1C_1$ - Intouch Triangle;
cA, cB, cC - Lucas circles of triangle $A_1B_1C_1$;
A_2, B_2, C_2 - Centers of the Lucas circles of triangle $A_1B_1C_1$;
c - Radical Circle of the Lucas circles of triangle $A_1B_1C_1$;
R - Center of the Radical Circle of the Lucas circles of triangle $A_1B_1C_1$;
The Gergonne Point Ge is the External Center of Similitude of the Outer Soddy Circle and the Radical Circle of the Lucas Circles of the Intouch Triangle.

Theorem 41. The Gergonne Point is the External Center of Similitude of the Adams Circle and the Lemoine Circle of the Intouch Triangle.

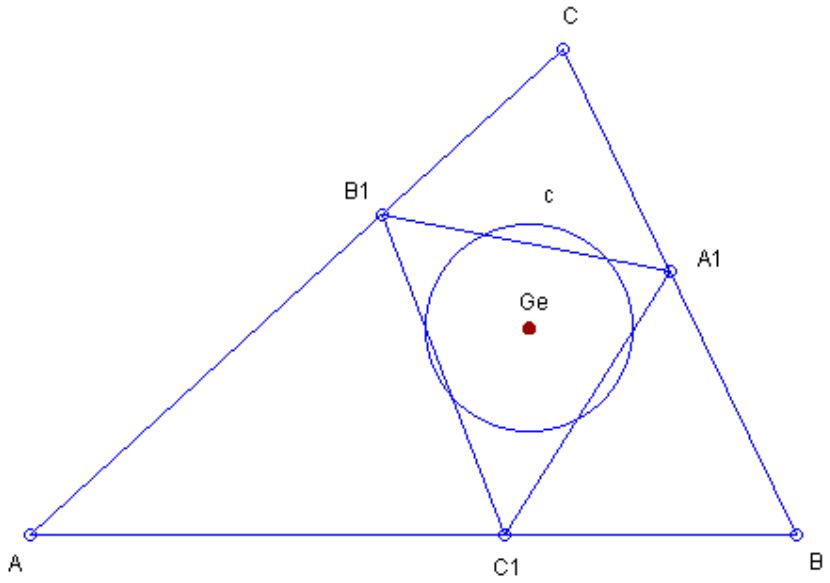
See the Figure:



Ge - Gergonne Point;
c - Adams Circle;
P - Center of the Adams Circle;
$A_1B_1C_1$ - Intouch Triangle;
c1 - Lemoine Circle of triangle $A_1B_1C_1$;
L - Center of the Lemoine Circle of triangle $A_1B_1C_1$;
The Gergonne Point Ge is the External Center of Similitude of the Adams Circle and the Lemoine Circle of the Intouch Triangle.

Theorem 42. The Gergonne Point is the Center of the Cosine Circle of the Intouch Triangle.

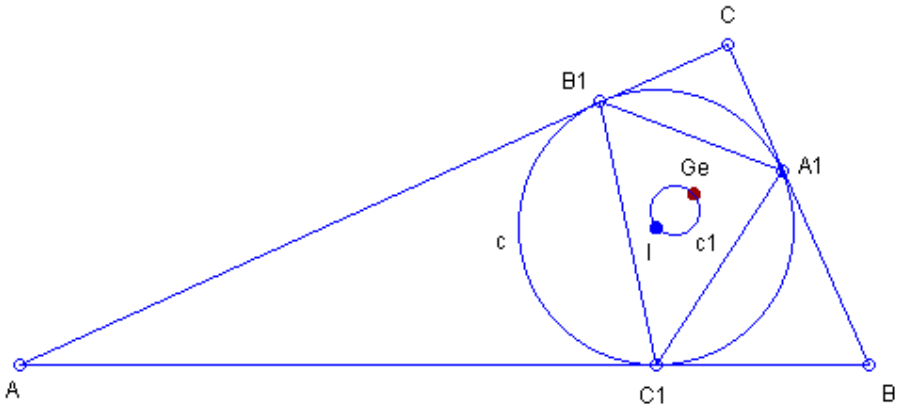
See the Figure:



Ge - Gergonne Point;
$A_1B_1C_1$ - Intouch Triangle;
c - Cosine Circle of triangle $A_1B_1C_1$;
The Gergonne Point Ge is the Center of the Cosine Circle of the Intouch Triangle.

Theorem 43. The Gergonne Point lies on the Brocard Circle of the Intouch Triangle.

See the Figure:



Ge - Gergonne Point;
c - Incircle;
I - Incenter;
$A_1B_1C_1$ - Intouch Triangle;
c1 - Brocard Circle of triangle $A_1B_1C_1$;
The Gergonne Point Ge lies on the Brocard Circle of the Intouch Triangle.

Theorems about Gergonne points of triangles

In different triangles the Gergonne points of these triangles could coincide with other named points. Here we give computer-generated theorems about the Gergonne points of different triangles.

1. The Gergonne Point of the Euler Triangle is the Midpoint of the Gergonne Point and the Orthocenter.
2. The Gergonne Point of the Medial Triangle of the Medial Triangle is the Complement of the Mittenpunkt.
3. The Gergonne Point of the Orthic Triangle of the Circum-Incentral Triangle is the Midpoint of the Gergonne Point and the Incenter.
4. The Gergonne Point of the Medial Triangle of the Euler Triangle is the Midpoint of the Mittenpunkt and the Orthocenter.
5. The Gergonne Point of the Tangential Triangle of the Excentral Triangle is the Mittenpunkt.
6. The Gergonne Point of the Tangential Triangle of the Circum-Incentral Triangle is the Midpoint of the Incenter and the Mittenpunkt.
7. The Gergonne Point of the Euler Triangle of the Medial Triangle is the Midpoint of the Circumcenter and the Mittenpunkt.
8. The Gergonne Point of the Johnson Triangle of the Medial Triangle is the Midpoint of the Circumcenter and the Gergonne Point.
9. The Gergonne Point of the Medial Triangle of the Orthic Triangle of the Intouch Triangle is the Gergonne Point.
10. The Gergonne Point of the Medial Triangle of the Orthic Triangle of the Circum-Incentral Triangle is the Midpoint of the Incenter and the Mittenpunkt.
11. The Gergonne Point of the Tangential Triangle of the Excentral Triangle of the Medial Triangle is the Complement of the Mittenpunkt.
12. The Gergonne Point of the Tangential Triangle of the Excentral Triangle of the Anticomplementary Triangle is the Gergonne Point.
13. The Gergonne Point of the Tangential Triangle of the Excentral Triangle of the Euler Triangle is the Midpoint of the Mittenpunkt and the Orthocenter.
14. The Gergonne Point of the Anticomplementary Triangle of the Euler Triangle of the Medial Triangle is the Midpoint of the Circumcenter and the Gergonne Point.

We invite the reader to select the new theorems and to prove them.

Thanks

The figures in this note are produced by using the program C.a.R. (Compass and Ruler), an amazing program created by Rene Grothmann. The Grothmann's program is available for download at the Web. It is free and open source. Many thanks to Rene Grothmann for his wonderful program.

I would like to thank to persons who contributed via personal correspondence for clarification of the state of the art in the area of automated mathematical discovery: Prof. Tobias Nipkow, Technical University of Munich; Prof. Bruno Buchberger, Research Institute for Symbolic Computation, Austria; Prof. Xiao-Shan Gao, Institute of Systems Science, Beijing, Prof. Alan Bundy, University of Edinburgh, Prof. Geoff Sutcliffe, University of Miami; Prof. Dongming Wang, University of Paris VI and Beihan University,

Beijing; and Prof. Simon Colton, Imperial College, London.

I would like to thank to persons who supported my research during the years: Prof. Dimiter Skordev, Prof. Ivan Dimovski, Prof. Grozyo Stanilov, Prof. Stancho Dimiev, Prof. Petar Kenderov, Prof. Ljubomir Ivanov, Prof. Nikolay Yanev, Prof. Borislav Lazarov, Prof. Radoslav Pavlov, Prof. Kalcho Todorov, and many others.

References

1. Simon and Newell, Heuristic problem solving: The next advance in operations research, Operations Research, 6(1) (1958) 1–10.
2. D. Dekov, Computer-Generated Encyclopedia of Euclidean Geometry, First Edition, 2006, http://www.dekovsoft.com/e1/.
3. D. Dekov, papers in the Journal of Computer-Generated Euclidean Geometry, 2006, 2007, 2008, available at the Web, http://www.dekovsoft.com/j/.
4. Eric W. Weisstein, MathWorld - A Wolfram Web Resource, http://mathworld.wolfram.com/.

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Preprint: 12 March 2008
Publication Date: 10 March 2009