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Learning through discoveries

Sava Grozdev, Deko Dekov

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Abstract. This paper describes Discoverer, a program for learning and teaching geometry with the help of a computer. The program is designed to discover new theorems in Euclidean geometry. The results are expressed in natural language. Discoverer is specially suited to be used as a learning tool in high schools and universities.

Keywords: Human–computer interface; Interactive learning environments; Media in education; Dynamic geometry..

1. Introduction

Dynamic geometry systems like Cabri or C.a.R. and computer algebra systems such as Maple or MatLab have highly influenced high school and university education in the area of Euclidean geometry. See e.g (Botana & Valcarce, 2002), (Hašek, 2013). But we have to conclude that there is still a missing component – a computer program, able to discover new theorems in Euclidean geometry. The computer program "Discoverer" is designed to fill this gap.

The computer program "Discoverer", created by the authors, is designed to discover new theorems in Euclidean geometry. "Discoverer" is an artificial intelligence system for discovery of new knowledge. The prototype of the computer program has discovered a few thousands new theorems in Euclidean geometry, and now a result discovered by the prototype of the "Discoverer" even is quoted in the Wikipedia (Wikipedia, Yff Center of Congruence). The authors expect that the "Discoverer" will be available for users staring with September 2014.

Below we will give an example about the use of "Discoverer".

2. Remarkable points of the triangle

A complete list or remarkable points of the triangle, available in the literature, is given in the Kimberling's encyclopedia of Triangle Centers ETC (Kimberling, ETC).

Suppose we want to discover new theorems for midpoints of remarkable points. In many cases, the midpoints of the remarkable points are again remarkable points. In order to investigate this topic, we may proceed as follows.

We select a few remarkable points from the database of “Discoverer”. In this example, let us select the first ten points from the Kimberling’s list (Kimberling). Then the “Discoverer” finds the midpoints of the selected points and produces a list with the midpoints. The list is enclosed to this paper as the List P. Then the “Discoverer” may apply the procedure “Partial Identification of Points”. Note that “Discoverer” possesses similar procedures for all kinds of geometrical objects. The procedure produces three lists and two tables. List K contains all points from List P, which are available in the Kimberling’s ETC. List D contains the rest of List P. In list P-X to any point from the List K is assigned a point from the Kimberling’s ETC. Tables P-X and X-P rewrite list P-X in two kinds of tabular forms. The files with these lists are enclosed. Note that “Discoverer” produces results in HTML-format, so that the results are ready for publication in the Web.

We may want to find new theorems about points from list K. These points are available at the Kimberling’s ETC, but the “Discoverer” finds a number of new theorems, that is theorems, not available in (Kimberling). Let us look at list D. If a point is in this list, it is not available in (Kimberling), hence we may expect that no theorem is available in the literature about this point.

3. The new theorems

Let us select from list D the “Midpoint of the Circumcenter and the Nagel Point” (no 5 in the list). Then “Discoverer” may apply the procedure “Roles of a Point”. Note that “Discoverer” has similar procedures for all kinds of geometrical objects. The procedure produces the list R–P (Roles of Points), containing theorems about the selected object. Since the selected point is from list D, we may expect that the theorems from list R–P are new ones, not available in the literature.

Let us consider the theorems from list R–P. Let us select the following theorem:

Theorem 1. (List R–P, no.19) The Midpoint of the Circumcenter and the Nagel Point is the Complement of the Reflection of the Circumcenter in the Incenter.

Now we may use a program for dynamic geometry, like C.a.R., and a program for symbolic computations such as Maple, in order to clarify the produced theorem. By using a dynamic geometry system, we may draw a picture, and also we may visually verify the theorem. The proofs of “Discoverer” are not in the traditional format, but we may use Maple for easy composition of the proofs. The Maple file with the proof of theorem 1 is enclosed to this paper.

For the definitions related to theorem 1, see (Weisstein), or geometry textbooks for secondary schools. Recall two definitions. Given triangle *ABC*, let A_1 be the point at which the *A*-excircle meets the side *BC*, and define B_1 and C_1 similarly. Then the lines AA_1, BB_1 and CC_1 concur in a point, known as the *Nagel point*. See (Weisstein, Nagel point). If points M and R are collinear with the centroid *G*, then *M* is the *complement* of *R* if *G* trisects segment *MR* and is closer to *M* than to *R*. See (Kimberling, Glossary, Complement).

Figure 1 illustrates theorem 1. The figure is produced by using C.a.R., a nice program for dynamic geometry, created by Rene Grothmann. Point *O* is the circumcenter, point *Na* is the Nagel point, point *M* is the midpoint of *O* and *Na*, point *I* is the incenter, point *G* is the centroid, and point *R* is the reflection of the circumcenter in the incenter. Then point *M* is the complement of point *R*.

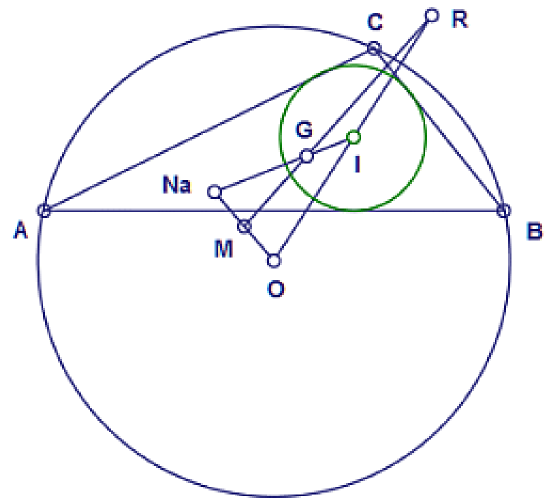

Fig.1. Point *M* is the complement of point R.

Note that if we want to announce a new remarkable point of the triangle, it is desirable we to announce the barycentric coordinates of this point. (In the Kimberling's encyclopedia this is a requirement). With the help of Maple it is easy we to find the barycentric coordinates of the points. Since in the proofs of theorem 1 we use barycentric coordinates, we obtain as fragments of the proof the barycentric coordinates of the point "Midpoint of the Circumcenter and the Nagel Point". In addition, a separate Maple file containing the calculations of the barycentric coordinates is enclosed.

4. From roles of objects to new theorems

Given a list of roles of a geometrical object, we may take two of the roles in order to compose a new theorem. In the above example, let us consider two of the roles from list R–P, e.g. no.3 and 23. We may compose the following theorem:

Theorem 2. (List R–P, no.3 and 23) The Reflection of the Nine-Point Center in the Spieker Center is the Bevan Point of the Euler Triangle of the Center of the Fuhrmann Circle.

We may expect that the above theorem is a new one, since the roles in the list R–P are not included in (Kimberling). Below we use C.a.R. for clarification and visual verification of theorem 2.

Given triangle *ABC*, note that the *Euler triangle of a point P* is the triangle whose vertices are the midpoints of the segments *AP*, *BP* and *CP*. The *nine-point center* is the center of the circumcircle of the medial triangle (Weisstein, Nine-point Center), the *Spieker center* is the center of the incircle of the medial triangle

(Weisstein, Spieker center), and the *Bevan point* is the center of the circumcircle of the excentral triangle, that is, center of the circle circumscribed about the triangle whose vertices are the centers of the excircles (Weisstein, Bevan point). The *Fuhrmann triangle* is the triangle formed by reflecting the midpoints of the arcs *BC*, *CA* and *AB* of the circumcircle, opposite to angles *A*, *B* and *C*, respectively, about the lines *BC*, *CA*, and *AB*, respectively. See (Weisstein, Fuhrmann Triangle) The *Fuhrmann circle* is the circumcircle of the Fuhrmann triangle. See (Weisstein, Fuhrmann Circle).

Figure 2 illustrates theorem 2. Point *M* is the reflection of the nine-point center in the Spieker center, point *P* is the circumcenter of the Fuhrmann triangle, $\triangle DEF$ is the Euler triangle of point *P*, points *Ja*, *Jb* and *Jc* are the centers of the excircles of $\triangle DEF$. Now point *M* is the circumcenter of $\triangle JaJbJc$.

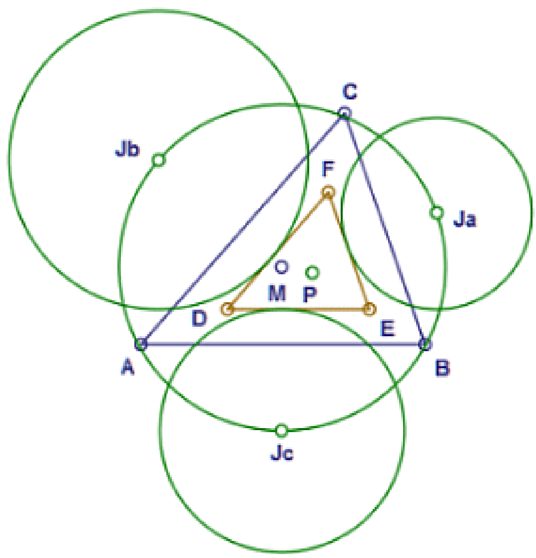

Fig.2. Point *M* is the circumcenter of $\triangle JaJbJc$.

List R-X contains roles of one of the points from list D. In a similar manner, “Discoverer” may produce lists of roles for the other points from list D. Moreover, we may use additional procedures for finding the lines, circles, etc. which contain the point.

5. The capabilities of “Discoverer”

In the above example we have selected from the database of “Discoverer” as starting point the set, containing the first ten points in (Kimberling). We may select from the database of “Discoverer” a larger set, and by this way we may obtain larger set of results. Currently the database of “Discoverer” contains approximately 500 000 points. At present, this is the biggest database of remarkable points of the triangle. The authors expect that the final version of “Discoverer” will contain about 5 millions remarkable points.

The “Discoverer” contains also large databases with other remarkable objects, like remarkable triangles, circles, lines, triads of circles, etc. Hence, we may find new theorems about different kind of geometrical objects.

In the above example we have selected one of the operations in Euclidean geometry – to find the midpoint of two points. In the “Discoverer” are implemented hundreds geometry operations, so that we have a rich choice. A number of procedures, similar to the above used procedures are available. The “Discoverer” procedures are applicable for different kinds of objects.

“Discoverer” may discover different kinds of theorems. See (Grozdev & Dekov, 2013a). One of capabilities of the computer program is to discover remarkable points which lie on remarkable curves. Note that “Discoverer” has also the capability easily to solve extremal problems in the geometry of triangle. The capability of “Discoverer” to investigate the construction problems (compass-and-ruler problems) by using a sequence of questions and answers, are discussed in (Grozdev and Dekov, 2013b), where a new solution to the classical Malfatti circles construction problem is given. We may expect that “Discoverer” may easily discover about 100 000 valuable theorems, and a few millions not so valuable theorems in Euclidean geometry.

Note that “Discoverer” is not only the first computer program able easily to discover new theorems in mathematics, but it seems that it is the first program, able easily to produce new knowledge in science. By using the principles of “Discoverer”, a number of similar computer programs may be created for the areas of high school teaching (e.g. physics, chemistry, etc.). This will give the opportunity the teachers to activate the interest of the students to their areas. By this way, a new direction in the teaching and learning would be introduced – learning through discoveries.

6. Conclusions

The computer program “Discoverer” is a useful complement to the set of tools for investigations in Euclidean geometry. The program fills a gap in the existing set of tools .It provides the possibility the students easily to discover new theorems, and by the help of the other tools, to clarify the theorems. Since the theorems are new ones, the students are capable even to submit the results as a discovery. But they have to mention the role of the “Discoverer” in their investigations. The authors hope that by joint efforts of the authors and the users, it will be produced also computer-generated encyclopedia of Euclidean geometry, designed for high schools and universities.

Appendix

The file 2014-1.zip is enclosed. The file contains the files quoted in this paper. Seven of the enclosed files are in HTML-format, and 2 files are in the Maple MWS-format.

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Sava Grozdev, Sofia, Bulgaria, sava.grozdev@gmail.com

Deko Dekov, Stara Zagora, Bulgaria, ddekov@ddekov.eu