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A New Relation between the Steiner Circumellipse and the Kiepert Hyperbola

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Abstract. The authors present a new relation between the points on the Steiner circumellipse and points on the Kiepert hyperbola. The result is discovered by the computer program "Discoverer", created by the authors.

During the years a number of notable points of the triangle have been discovered to lie on the Steiner circumellipse. Eric Weisstein [3] has presented a list of 19 points, which lie on the Steiner circumellipse.

In 2000, Paul Yiu [4] introduced the barycentric products (in this note we use the term "product") of points and gave a number of applications. Theorem 1 below gives one additional application of the products. Theorem 1 is discovered by the computer program "Discoverer", created by the authors. The proof of "Discoverer" is non-standard. Below we present a proof which is rewritten in a traditional manner.

Theorem 1 presents an unexpected result, which shows that we can obtain a new point on the Steiner circumellipse, if we start from an arbitrary point which lies on the Kiepert hyperbola.

We use barycentric coordinates [2,3,4,5] with respect to the reference triangle *ABC*. We denote by a,b,c the side lengths of triangle *ABC*, $a = BC$, $b = CA$ and $c = AB$. Given two points $P = (u_1 : v_1 : w_1)$ and $Q = (u_2 : v_2 : w_2)$, the product of P and Q is the point $P.Q = (u_1u_2 : v_1v_2 : w_1w_2)$. The Steiner point has barycentric coordinates [2]:

$$X(99) = \left(\frac{1}{b^2 - c^2} : \frac{1}{c^2 - a^2} : \frac{1}{a^2 - b^2}\right). \qquad (1)$$

The Kiepert hyperbola consists of the Kiepert perspectors [6]:

$$K(t)=\left(\frac{1}{S_A+t}:\frac{1}{S_B+t}:\frac{1}{S_C+t}\right), \qquad (2)$$

where $t\in\mathbb{R}$ and S_A,S_B,S_C are the Conway's notations [5]:

$$S_A=\frac{b^2+c^2-a^2}{2},\quad S_B=\frac{c^2+a^2-b^2}{2}\quad\text{and}\quad S_C=\frac{a^2+b^2-c^2}{2}$$

The barycentric equation of the Steiner circumellipse of triangle ABC is as follows [5]:

$$yz+zx+xy=0. \qquad (3)$$

Theorem 1. *The product of the Steiner point and any Kiepert perspector lies on the Steiner circumellipse.*

Proof. We form the product of the Steiner point (1) and the Kiepert perspector (2). Denote the barycentric coordinates of the product by p,q,r. We substitute in the equation of the Steiner circumellipse (3) the coordinates p,q,r for the unknowns x,y,z, respectively, and after simplification, we obtain the identity $0=0$. This proves the theorem.

If we want to avoid calculations by hand in the above proof, we may use a computer algebra system, like Maple. Below we give the Maple commands (copy and paste in a Maple file). The barycentric coordinates of points in the proof are as follows: Steiner Point $=(uS:vS:wS)$, Kiepert Perspector = $(uK:vK:wK)$, Product of the Steiner Point and the Kiepert Perspector = $(p:q:r)$. The left-hand side of the equation (3) is denoted by *S.* We obtain as output $S:=0$. Theorem is proved.

```
S:=y*z+z*x+x*y;
uS:=1/(b^2-c^2);vS:=1/(c^2-a^2);wS:=1/(a^2-b^2);
SA:=(-a^2+b^2+c^2)/2;
SB:=(-b^2+c^2+a^2)/2;
SC:=(-c^2+a^2+b^2)/2;
uK:=1/(SA+t):uK:=simplify(uK);
vK:=1/(SB+t):vK:=simplify(vK);
wK:=1/(SC+t):wK:=simplify(wK);
p:=uS*uK;q:=vS*vK;r:=wS*wK;
S:=subs(x=p,y=q,z=r,S);
S:=simplify(S);
```

Figure 1 illustrates the theorem. In figure 1, point *S* is the Steiner point. We take an arbitrary Kiepert perspector *P* as a starting point, and by using compass and ruler, we construct point *X* = the product of *P* and *S* [4]. The point *X* lies on the Steiner circumellipse.

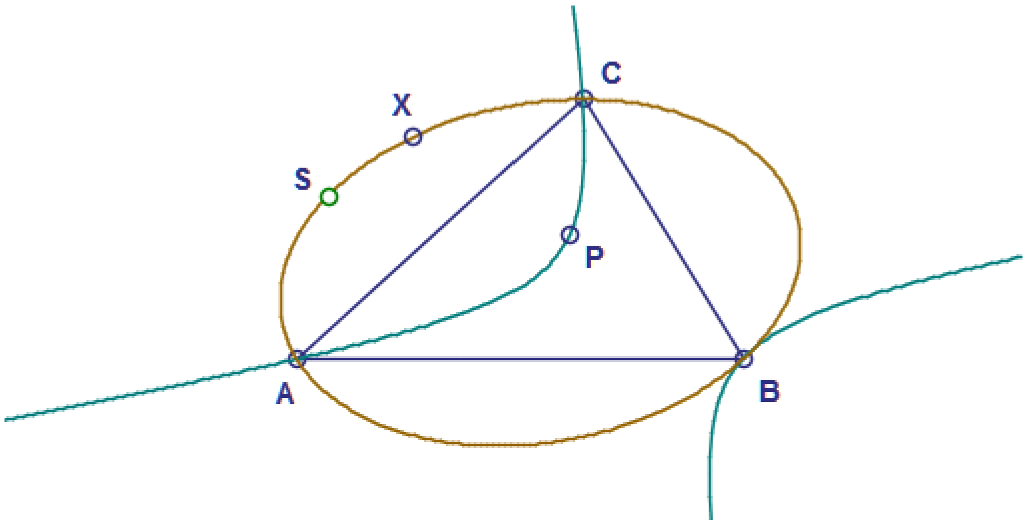

Fig. 1. Triangle *ABC*, its Kiepert hyperbola and its Steiner circumellipse.

The table below gives a few special cases of theorem 1.

	Product of the Steiner point and the	Kinbering [2] notation of the product
1	Centroid	X(99)
2	Orthocenter	X(648)
3	Spieker Center	X(190)
4	Third Brocard Point	X(670)
5	Tarry Point	X(2966)

The above theorem may be used as follows. We take the points which lie on the Kiepert hyperbola and form their products with the Steiner point. By this way we obtain a number of new points which lie on the Steiner circumellipse. In [1] the reader may see a list of 2387 notable points which lie on the Steiner circumellipse. The list is partially produced by using the above theorem. The list is produced by the computer program "Discoverer".

References

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