

The computer program “Discoverer” as a tool of mathematical investigation

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Abstract. This paper describes “Discoverer”, a program for learning and teaching geometry with the help of a computer. The program is designed to discover new theorems in Euclidean geometry. The results are expressed in natural language. Discoverer is specially suited to be used as a learning tool in high schools and universities.

1. INTRODUCTION

Dynamic geometry systems like Cabri or C.a.R. and computer algebra systems such as Maple or MatLab have highly influenced high school and university education in the area of euclidean geometry. See e.g (Botana & Valcarce, 2002), (Hašek, 2013). But we have to conclude that there is still a missing component – a computer program, able to discover new theorems in Euclidean geometry. The computer program “Discoverer” is designed to fill this gap. “Discoverer” is an artificial intelligence system for discovery of new knowledge. The prototype of the computer program, created in 2006, has discovered a few thousands new theorems in Euclidean geometry, and now a result discovered by the prototype of the “Discoverer” even is quoted in the Wikipedia (Wikipedia, Yff Center of Congruence).

In this paper we discuss the use of “Discoverer” in high school and university education. See also Grozdev and Dekov, 2013a-b. The definitions the reader may find in Grozdev and Dekov, 2014.

2. CIRCUMCONICS

We will give a few applications of “Discoverer” related to the investigation of circumconics.

Recall that a circumconic is a conic in the plane of the reference triangle ABC that passes through the vertices of the triangle. The *defining line* of a circumconic is the line, isogonal to the circumconic. The *defining point* of a circumconic is the point, isogonal to the perspector of the circumconic. Note that the defining line is the tripolar of the defining point. Given a circumconic, its *associated circumconic* is the circumconic whose perspector is the center of the given circumconic. Note that the Steiner circumconic is associated to itself. If two circumconics are associated, we say that their perspectors are *associated*.

A circumconic is defined, if it is given (See Yiu, 2001):

1. The perspector of the circumconic, or
2. The center of the circumconic, or
3. The defining line of the circumconic, or
4. The defining point of the circumconic, or
5. The equation of the circumconic in barycentric coordinates, or
6. The equation of the circumconic in Cartesian coordinates.

If the perspector of a circumconic is given, we could find (See Yiu, 2001):

- the center of the circumconic,
- the fourth intersection of the circumconic, that is, the intersection of the circumconic and the circumcircle, distinct from the vertices of the reference triangle (the first three intersections are the vertices of the reference triangle), and
- the fifth intersection, that is the intersection of the circumconic and its associated circumconic.

Table 1 gives a few pairs of associated points, provided the both points of the pair are included in the Kimberling’s Encyclopedia of Triangle Centers (ETC). We give also the numbers of the notable points, used in Kimberling’s ETC. The table is produced by computer program “Discoverer”. A few of the pairs in the table are available in the literature, possibly a few a new.

Note that the associated points of the majority of the notable points in Kimberling, are not available in Kimberling. For example, the associated points of the following points are not included in Kimberling: Outer Fermat Point, Inner Fermat Point, First Isodynamic Point, Second Isodynamic Point, Outer Napoleon Point, Inner Napoleon Point, Clawson Point, de Longchamps Point, Schiffler Point, Exeter Point, etc.

	Point	Associated point
1	X(1) Incenter	X(9) Mittenpunkt
2	X(2) Centroid	X(2) Centroid
3	X(3) Circumcenter	X(6) Symmedian Point
4	X(4) Orthocenter	X(1249)
5	X(5) Nine-Point Center	X(216)
6	X(7) Gergonne Point	X(3160)
7	X(8) Nagel Point	X(3161)
8	X(10) Spieker Center	X(37) Grinberg Point
9	X(11) Feuerbach Point	X(650) Center of the Stevanovic Circle
	X(32) Third Power Point	X(206)
10	X(39) Brocard Midpoint	X(141) Symmedian Point of the Medial Triangle
11	X(55) Internal Center of Similitude of the Incircle and the Circumcircle	X(5452)
12	X(56) External Center of Similitude of the Incircle and the Circumcircle	X(478)
13	X(100) Anticomplement of the Feuerbach Point	X(5375)
14	X(125) Center of the Jerabek Hyperbola	X(647)
15	X(2092) Danneels-Apollonius Perspector	X(960)

Table 1. Pairs of associated notable points.

3. MATHEMATICAL INVESTIGATION BY MEANS OF COMPUTER

We may use “Discoverer” in order to find notable points of the triangle which lie on circumconic with the incenter as perspector. It is known (see Yiu 2001, page 110) that the Mittenpunkt is the center of this circumconic. The “Discoverer” produces lists of notable points. The results are given in the enclosed files.

Note that the points, discovered by "Discoverer" to lie on the circumconic, depend on the shape (e.g. acute-angled or obtuse-angled) of triangle ABC. In many cases a point lies on the circumconic independently on the shape of triangle ABC. For each point the user has to find the shape of triangle ABC such that the point lies on the circumconic. The authors of the "Discoverer" intend to produce the corresponding module of the computer program, but at the present time the module is not ready.

We see that “Discoverer” has discovered 57 notable points which lie on the circumconic with the incenter as perspector. Fourteen of these points are included in Kimberling’s ETC, and the rest of 43 notable points are not included in this encyclopedia. For a few of the discovered points it is known that they lie on the investigated circumconic, e.g. Yiu (page 110) noticed that the anticomplement of the Feuerbach point is the fifth intersection of the circumconic. By using “Discoverer” we easily see that the

fourth intersection of the circumconic coincides with its fifth intersection.. The authors expect that for the majority of the discovered points the result is new. In addition, we may expect that the majority of the points, which are not included in Kimberling's ETC are not available in the literature.

In figure 1, c is the circumconic having the incenter I as perspector. The sidelines of triangle $JaJbJc$ are the tangents to the circumconic at the vertices of triangle ABC . Point M is the Mittenpunkt, the center of the circumconic. Point P is the fourth (and also the fifth) intersection of the circumconic. On the circumconic lie 56 notable points, besides point P . Two of these points are on figure 1. Point Q is the complement of the isotomic conjugate of the cyclocevian conjugate of the isogonal conjugate of the inverse of the incenter in the circumcircle (point X(88) in Kimberling's ETC), and point N is the product of the Clawson point and the Steiner point (point X(162) in Kimberling's ETC).

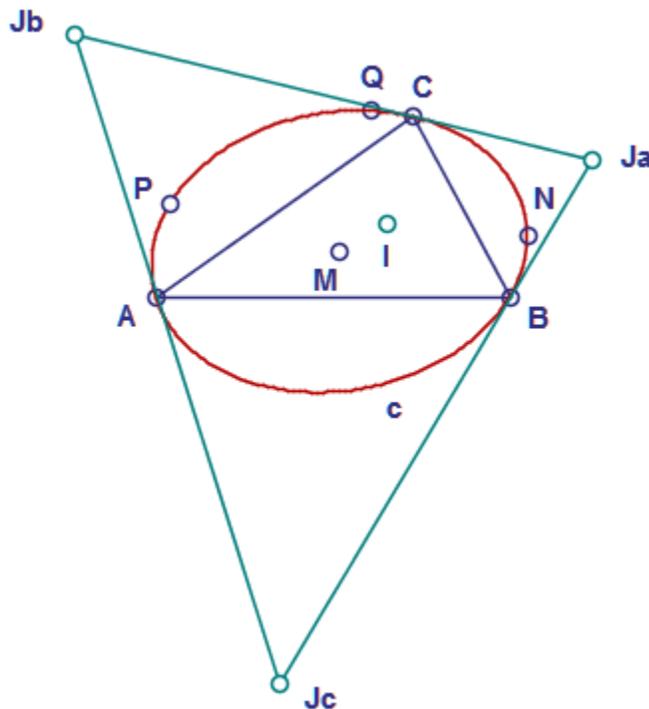


Fig. 1. The circumconic having the incenter as perspector.

Note that if we use a computer program for dynamic geometry, and if we want to construct a macro of a conic, we need five points which lie on the conic. In the case of circumconic we have already three points - the vertices of the reference triangle. Hence, we have to find two additional points. In many cases it is convenient we to use the fourth and fifth intersections of the circumconic.

The proofs of the computer program "Discoverer" are non-standard. Hence, at this stage, we have to present new proofs, which are written in a traditional manner. Below we present a proof of the following theorem, available in the enclosed files:

Theorem 1. *Quotient of the second power point divided by the center of the Parry circle lies on the circumconic with the incenter as perspector.*

Proof. We use barycentric coordinates. We denote by a , b and c the side lengths of triangle ABC , $a = BC$, $b = CA$ and $c = AB$. We recall that the barycentric equation of the circumconic with the incenter as perspector is as follows (see Yiu 2001):

$$ayz + bzx + cxy = 0$$

We form the quotient of the second power point and the center of the Parry circle. The barycentric coordinates of these points are given in Kimberling's ETC, points X(31) and X(351), respectively. Denote the barycentric coordinates of the quotient by p, q, r . We substitute in the above equation of the circumconic the coordinates p, q, r for the unknowns x, y, z , respectively, and after simplification, we obtain the identity $0 = 0$. This proves the theorem.

If we want to avoid calculations by hand in the above proof, we may use a computer algebra system, like Maple. The Maple file is enclosed to this paper as a supplementary material. Below we give the Maple commands (copy and paste in a Maple file). The barycentric coordinates of points in the proof are as follows: Second power point = (uP, vP, wP) , Center of the Parry circle = (uQ, vQ, wQ) , quotient $P/Q = (p : q : r)$. The left-hand side of the equation of the circumconic is denoted by E . We obtain as output $E := 0$. Theorem is proved.

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E:=a*y*z+b*z*x+c*x*y;
uP:=a^3;vP:=b^3;wP:=c^3;
uQ:=a^2*(b^2-c^2)*(b^2 + c^2 - 2*a^2);
vQ:=b^2*(c^2-a^2)*(c^2 + a^2 - 2*b^2);
wQ:=c^2*(a^2-b^2)*(a^2 + b^2 - 2*c^2);
p:=uP/uQ;q:=vP/vQ;r:=wP/wQ;
E:=subs(x=p,y=q,z=r,E);
E:=simplify(E);
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The reader is invited to prove the other statements of the paper. Note that it is reasonable we to calculate the barycentric coordinates of the points, which are not included in the Kimberlings'ETC. A sample calculation of the barycentric coordinates of the point "Quotient of the Second Power Point divided by the Center of the Parry Circle" is given in an enclosed Maple file.

The "Discoverer" also finds notable points which lie on the associated circumconic of the circumconic with the incenter as perspector. The results are given in the enclosed files, produced by "Discoverer".

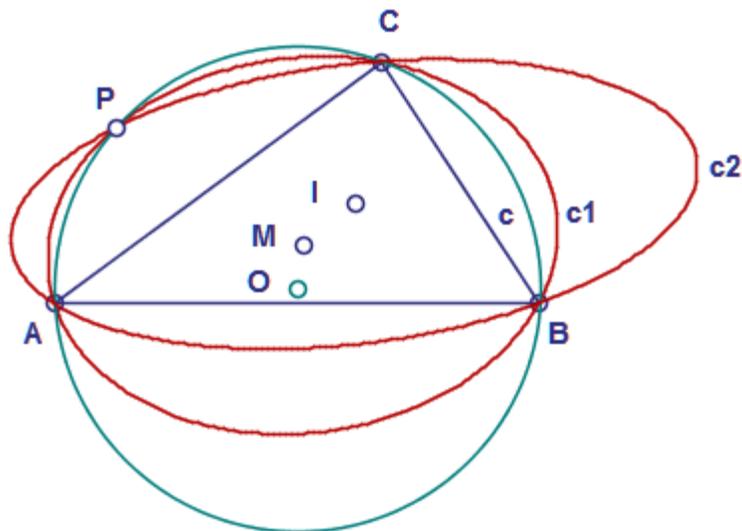


Fig. 2. The circumconic $c1$, having the incenter as Perspector, and its associated circumconic $c2$, having the Mittenpunkt as perspector.

Figure 2 illustrates these two circumconics. In figure 2, $c1$ is the circumconic with the incenter as perspector and Mittenpunkt as center, $c2$ is its associated circumconic, that is the circumconic with Mittenpunkt as perspector and the incenter as center, and c is the circumcircle of triangle ABC . Point P (point $X(100)$ in Kimberling's ETC) is the intersection of the circumconics. Point O is the circumcenter, I is the Incenter, M is the Mittenpunkt.

4. PREPARATION OF SCHOLARLY ESSAYS BY STUDENTS

The students could use "Discoverer" to investigate many topics in euclidean geometry. We call the use of "Discoverer" by the students for educational purposes "learning through discoveries". Below is a description of the procedure for learning through discoveries for the special case of investigation of circumconics. The procedure is as follows:

1. The student chooses a notable point in the plane of triangle ABC .
2. "Discoverer" calculates the center, the fourth and the fifth intersections of the circumconic having the selected notable point as perspector. Note that the student could calculate these points by hand.
3. "Discoverer" discovers a list of notable points which lie on the circumconic. The help of "Discoverer" at this stage is essential.
4. "Discoverer" discovers which of the above discovered points are included in Kimberling's database. The help of "Discoverer" at this stage is essential.
5. For any point, the student finds the shape of triangle ABC such that the point lies on the circumconic. For this purpose the student uses the computer programs C.a.R. and Maple.

6. “Discoverer” discovers new properties of the points which are not available in the Kimberlings’s ETC. The help of “Discoverer” at this stage is essential.
7. The student uses computer algebra system, like Maple in order to find the barycentric coordinates of the new points, that is, the points which are not available at the Kimberling’s ETC.
8. The student uses the computer program for dynamic geometry, like C.a.R., in order to investigate the relation between the results and the shapes of the triangles. The student investigates the ruler and compass constructions of the new points, produces macros for these points and animations for the constructions. The student produces a macro for the investigated circumconic. The student produces also computer graphics for his or her essay.
9. The student uses Maple in order to prepare proofs of the statements of the essay.
10. The student may investigate also another problems related to the circumconics, by using the “Discoverer”, C.a.R. and Maple. The student may investigate the circumconic by using cartesian coordinates. This item is especially suitable for students, who take the course of Analytic Geometry.
11. The student prepares the scholarly essay. The essay contains as supplementary material the HTML-files, produced by “Discoverer”, the C.a.R. files and graphics and the Maple files, produced by the student.
12. The barycentric coordinates of the new points may be submitted for publication in the Kimberling’s encyclopedia. The student has to submit his or her essay for publication in the Journal of Computer-Generated Mathematics, www.ddekov.eu/j/, and in the Computer-Generated Encyclopedia of Euclidean Geometry, eg-enc.webege.com.

Note that the work of the student could be highly facilitated, if the student uses a collection of macros for C.a.R. and a collection of commands for Maple. The Maple collection may contain commands for production of the points, triangles, circles, etc, and commands for operations on these objects. If the student possesses such a collection, he may produce easily the proofs of the theorems just by copy and paste. This approach will give to the students the possibility to concentrate their attention on the algorithms of the proofs. Preparation of a suitable collection of Maple commands is an interesting topic for scholarly essay.

During the work on the scholarly essay, the student will improve his skills to use a discovery system, like “Discoverer”, a computer algebra system like Maple, and a system for dynamic geometry, like C.a.R. (or GeoGebra). These skills are between the basic ones which the students have to master and improve during their education.

The “Discoverer” is able to help in many directions of Euclidean geometry. The scholarly essays of students are a power tool for improvement of education. Well written essays would be worth even for publication in the scientific journals. Also, it is reasonable the high schools and universities to begin publication of on-line journals,

collecting the essays of the students. Also, there are topics in euclidean geometry which are suitable for bachelor's and master's theses.

5. CONCLUSIONS

The computer program “Discoverer” is a useful complement to the set of tools for investigations in euclidean geometry. The program fills a gap in the existing set of tools .It provides the possibility the students easily to discover new theorems, and by the help of the other tools, to clarify the theorems. Since the theorems are new, the students are capable even to submit the results as a discovery. But they have to mention the role of the “Discoverer” in their investigations. Hopefully, the joint efforts of the authors and the users will produce also computer-generated encyclopedia of euclidean geometry, designed for high schools and universities.

Supplementary material

The file “2014-5.zip” is enclosed. It contains the files quoted in this paper.

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