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Supplementary Material to the Note by Grozdev and Dekov, published in Mathematical Gazette in November 2014

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Abstract. The aim of this note is to give to the reader the possibility to download the Maple files containing the proofs of the theorem and examples, given in [1].

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The aim of this note is to give to the reader the possibility to download the Maple files containing the proofs of the theorem and examples, published in [1]. We also give additional examples to the theorem published in [1]. The Maple files to the theorems and examples are available in the enclosed file “2014-6.zip”.

In [1] is published the following theorem:

Theorem 1. *Let P and Q be points, neither lying on a sideline of triangle ABC. If P and Q are isogonal conjugates with respect to ABC, then the Ceva product of their complements lies on the Kiepert hyperbola.*

Special cases of the theorem 1 are given in Table 1.

	P	$\tilde{P} * \tilde{Q}$
1	Incenter = (a,b,c)	Spieker Center = $(b+c, c+a, a+b)$
2	Centroid = $(1,1,1)$	Isotomic Conjugate of the Complement of the Complement of the Symmedian Point = $[(a^2+2b^2+c^2)(a^2+b^2+2c^2),$ $(2a^2+b^2+c^2)(a^2+b^2+2c^2),$ $(2a^2+b^2+c^2)(a^2+2b^2+c^2)]$

3	Circumcenter = $[a^2(b^2+c^2-a^2),$ $b^2(c^2+a^2-b^2),$ $c^2(a^2+b^2-c^2)]$	Isogonal Conjugate of the Center of the Taylor Circle. See the enclosed Maple file for the barycentric coordinates.
4	Steiner Point = $\left(\frac{1}{b^2-c^2},\frac{1}{c^2-a^2},\frac{1}{a^2-b^2}\right)$	Orthocenter = $[(a^2-b^2+c^2)(a^2+b^2-c^2),$ $(b^2-c^2+a^2)(b^2+c^2-a^2),$ $(c^2-a^2+b^2)(c^2+a^2-b^2)]$
5	Isogonal Conjugate of the Anticomplement of the Feuerbach Point = $[a(b-c),b(c-a),c(a-b)]$	Orthocenter

Table 1

The first three special cases in Table 1 are given in [1].

We follow the tradition in triangle geometry to give names of flowers of the transforms. Hence, we call "Lily transform" the transform $P \mapsto \tilde{P}*\tilde{Q}$. We denote $Lily(P)=\tilde{P}*\tilde{Q}$.

We can obtain additional special cases, if we use the following obvious proposition:

Proposition. If P and Q are isogonal conjugates wrt $\triangle ABC$, then their Lily transform coincide.

Corollaries to the above proposition: Lily(Symmedian Point) = Lily(Centroid), Lily(Orthocenter) = Lily(Circumcenter), etc.

Row 4 in Table 1 states that Lily(Steiner Point) = Orthocenter. We suggest to the reader to prove the following more general statement:

Hypothesis. Lily(P) = Orthocenter, if P lies on the circumcircle of $\triangle ABC$.

If we combine the above proposition and the above hypothesis, we obtain:

Statement. Lily(P) = Orthocenter, if $P=(u,v,w)$ is a point at Infinity and $u,v,w \neq 0$.

Te isogonal conjugate of the Incenter is equal to the Incenter. There are three additional points P such that the isogonal conjugate of P is equal to P. These points are given in Table 2. Note that the A-excenter coincides with the point with barycentric coordinates $(a,-b-c)$.

	P	$\tilde{P} * \tilde{Q}$
1	A-Excenter = $(-a, b, c)$	$(b+c, c-a, -a+b)$
2	B-Excenter = $(a, -b, c)$	$(-b+c, c+a, a-b)$
3	C-Excenter = $(a, b, -c)$	$(b-c, -c+a, a+b)$

Table 2

The points in Table 2, as well as the points in Table 3, are not triangle centers. For the definition of a triangle center see e.g. [2]. But we have to use these points, if we want to investigate the points on the Kiepert hyperbola.

	P	$\tilde{P} * \tilde{Q}$
1	(a, a, b)	See the enclosed Maple file for the barycentric coordinates.
2	First Brocard Point = $\left(\frac{1}{b^2}, \frac{1}{c^2}, \frac{1}{a^2}\right)$	See the enclosed Maple file for the barycentric coordinates.
3	$(1,1,2)$	$[3(3a^2+5b^2+c^2)(a^2+b^2+c^2),$ $3(a^2+b^2+c^2)(5a^2+3b^2+c^2),$ $(5a^2+3b^2+c^2)(3a^2+5b^2+c^2)]$
4	$(1,-2,3)$	$[(3a^2-c^2)(5a^2+3b^2-6c^2),$ $(5a^2+3b^2-c^2)(9a^2-6b^2-c^2),$ $(9a^2-6b^2-c^2)(3a^2-c^2)]$
5	$(-7,11,19)$	See the enclosed Maple file for the barycentric coordinates.

Table 3

We can omit the text "*Let P and Q be points, neither lying on a sideline of triangle ABC*" from the statement of theorem 1, if we make two changes.

First, we have to use generalized definition of the isogonal conjugate The generalized definition is as follows (See [3]): The isogonal conjugate of a point $P=(u,v,w)$ is the point (a^2vw, b^2wu, c^2uv).

Secondly, we have to add point (0,0,0) to the space which we use for calculations. For calculations we use the projective space $P^3(\mathbb{R})$ with basis A(1,0,0), B(0,1,0) and C(0,0,1). It is well known that $(0,0,0) \notin P^3(\mathbb{R})$. (See [4], paragraph 50, page 153). Hence we have to add (0,0,0) to $P^3(\mathbb{R})$.

If we accept the above two changes, we obtain:

Theorem 2. *If P and Q are isogonal conjugates with respect to ABC, then the Ceva product of their complements lies on the Kiepert hyperbola.*

The Maple file with the proof of theorem 2 is included in the enclosed file "2014-06.zip". Table 4 gives special cases to theorem 2.

	P	$\tilde{P} * \tilde{Q}$
1	(0,0,0)	(0,0,0)
2	Vertex A = (1,0,0)	(0,0,0)
3	Vertex B = (0,1,0)	(0,0,0)
4	Vertex C = (0,0,1)	(0,0,0)
5	Finite point on sideline BC = (0,v,w)	Centroid = (1,1,1)
6	Finite point on sideline CA = (u,0,w)	Centroid = (1,1,1)
7	Finite point on sideline AB = (u,v,0)	Centroid = (1,1,1)
8	(0,1,-1)	(0,0,0)
9	(1,0,-1)	(0,0,0)
10	(1,-1,0)	(0,0,0)

Table 4

Note about compass and ruler constructions. In figure 1, point *S* is the Steiner point, *c* is the circumcircle, point *gS* is the isogonal conjugate of *S*, points *cgS* is the complement of the isogonal conjugate of *S*, *L* is the line at Infinity (drawn in the finite space for our convenience), *R* is the Ceva product of points *cS* and *cgS*. The hyperbola in Fig.1 is the Kiepert hyperbola. We see that point *R* lies on the Kiepert hyperbola. Points *S* and *R*, the argument and the image of the Lily transform of point *S*, are finite points, but in order to obtain point *R* we have to go through the line at Infinity. If we restrict ourselves to the finite points, we will not be able to obtain point *R*. Hence, it is suitable we to include points at Infinity in our considerations. In many cases we will be able to construct the Lily transform of a finite point by using compass and ruler, but in many case we have to perform calculations, if we restrict ourselves to the classical compass and ruler.

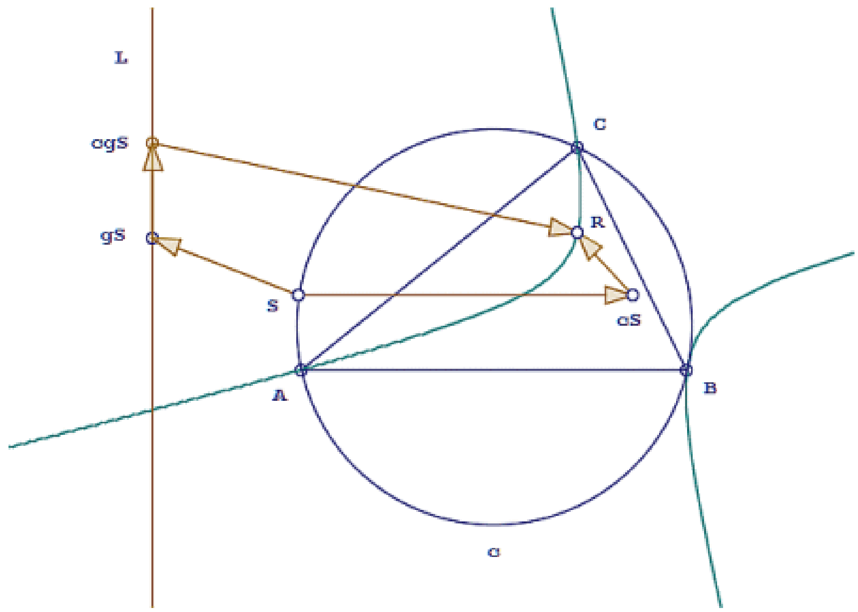

Fig. 1.

Enclosed file

The file “2014-6.zip” is enclosed. The file contains the Maple files quoted in this paper.

Acknowledgement

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References

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