Problem 97, published in the MathProblems, 2014, vol. 4, no 2, 263-264.

Solution to Problem 97

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We will use barycentric coordinates. Recall some formulas (see e.g. [3]). Given points $U = (u, v, w)$ and $P = (p, q, r)$. The vertices of the cevian triangle of U are the points $U_a = (0, v, w), U_b = (u, 0, w), U_c = (u, v, 0)$. The normalized barycentric coordinates of U are $(\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w})$. The reflection of U in P (U and P must be in normalized barycentric coordinates, see [1], section 3.1) is the point

(1)
$R = ((p-q-r)u+2p(v+w), (q-p-r)v+2q(u+w), (r-p-q)w+2r(u+v)).$

The equation of a line L through the points U and P (not necessary in normalized coordinates, see [3], section B1) is

(2) $$L : (vr - wq)x + (wp - ur)y + (uq - vp)z = 0.$$

Three lines $p_ix + q_iy + r_iz = 0,\ i = 1, 2, 3$, are concurrent (see [3], section B1) if and only if

(3) $$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

Denote by a, b and c the side lengths of $\triangle ABC$, $a = BC, b = CA$ and $c = AB$. As in [2], X(n), $n \in \mathbb{N}$, denotes a notable point.

In this problem U is the Nagel point of $\triangle ABC$, point X(8),

$$U = (b + c - a, c + a - b, a + b - c)$$

and P is the Spieker center of $\triangle ABC$, point X(10),

$$P = (b + c, c + a, a + b)$$

Denote by R_a, R_b and R_c the reflections of the vertices of the cevian triangle of U, that is points U_a, U_b and U_c in point P, respectively. By using (1), we obtain

$$\begin{aligned} R_a &= (2a(b + c), a^2 + b^2 - c^2, a^2 + c^2 - b^2), \\ R_b &= (a^2 + b^2 - c^2, 2b(a + c), b^2 + c^2 - a^2), \\ R_c &= (a^2 + c^2 - b^2, b^2 + c^2 - a^2, 2c(a + b)). \end{aligned}$$

Denote by L_a, L_b and L_c the lines AR_a, BR_b and CR_c, respectively. By using (2), we obtain the equtions of these lines:

$$\begin{array}{ll} L_a: & (b^2 - a^2 - c^2)y + (a^2 + b^2 - c^2)z = 0, \\ L_b: & (b^2 + c^2 - a^2)x + (c^2 - a^2 - b^2)z = 0, \\ L_c: & (a^2 - b^2 - c^2)x + (a^2 + c^2 - b^2)y = 0. \end{array}$$

By using (3), we obtain that the lines L_a, L_b and L_c concur in a point. The problem is solved.

After some additional work, we may identify the point of intersection of these lines, that is, the Prasolov product of the points. In this problem, the Prasolov product is the orthocenter of $\triangle ABC$.

Note that if we want to avoid calculations by hand, we may use a computer algebra system, like Maple. Once a Maple files is created, it could be used as a template for solving the Prasolov problem for any two points U and P whose barycentric coordinate are known.

Additional problems for the reader: By using the Maple template (or by calculations by hand) and the barycentric coordinates of points, given in [2], prove that the Prasolov products exist for the pairs of points given in the table below. In the table we also give the Prasolov products. Recall that the anticomplement of a point P is the image of the homothety with center the centroid and ratio -2. In the table aaP denotes the anticomplement of the anticomplement of point P. The Maple files with the solutions of the problems given below are available in the enclosed Supplementary Material.

We note that Problems 94, 97 and 109, published in 2014 in the journal MathProblems, as well as the solutions of these problems, are discovered by the computer program "Discoverer" created by the authors.

	U	P	Prasolov product
1	Orthocenter, X(4)	Taylor center, X(389)	X(3)
2	Symmedian point, X(6)	Brocard midpoint X(39)	X(76)
3	Nagel point, X(8)	Mittenpunkt, X(9)	X(7)
4	Kosnita point, X(54)	Taylor center, X(389)	X(52)
5	Centroid, X(2)	Arbitrary point P	aaP

Enclosure: Supplementary Material.

References

[1] Douillet, P. (2012). Translation of the Kimberlings Glossary into barycentrics, `http://eg-enc.webege.com/htm/links/glossary.pdf`

[2] Kimberling, C. Encyclopedia of Triangle Centers,
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[3] Schindler, M. and Cheny, E. (2012). Barycentric Coordinates in Olympiad Geometry. http://www.artofproblemsolving.com/Resources/Papers/Bary_full.pdf