Solition

We will use barycentric coordinates. Recall some formulas (see e.g. [3]). Given points $U = (u, v, w)$ and $P = (p, q, r)$. The vertices of the anticevian triangle of U are the points $U_a = (-u, v, w), U_b = (u, -v, w), U_c = (u, v, -w)$. The normalized barycentric coordinates of U are $(\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w})$. The reflection of U in P (U and P must be in normalized barycentric coordinates, see [1], section 3.1) is the point

(1)

$R = ((p-q-r)u+2p(v+w), (q-p-r)v+2q(u+w), (r-p-q)w+2r(u+v)).$

The equation of a line L through the points U and P (not necessary in normalized coordinates, see [3], section B1) is

(2) $$L : (vr - wq)x + (wp - ur)y + (uq - vp)z = 0.$$

Three lines $p_i x + q_i y + r_i z = 0,\ i = 1, 2, 3$, are concurrent (see [3], section B1) if and only if

(3) $$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

Denote by a, b and c the side lengths of $\triangle ABC$, $a = BC, b = CA$ and $c = AB$. As in [2], X(n), $n \in \mathbb{N}$, denotes a notable point.

In this problem $U = (a, b, c)$ is the incenter of $\triangle ABC$, point X(1), and $P = (b+c, c+a, a+b)$ is the Spieker center of $\triangle ABC$, point X(10), Denote by R_a, R_b and R_c the reflections of the vertices of the anticevian triangle of U, that is points U_a, U_b and U_c in point P, respectively. By using (1), we obtain

$$\begin{aligned} R_a &= (b^2 + 2bc + c^2 + a^2, c^2 - a^2 - b^2, b^2 - a^2 - c^2), \\ R_b &= (c^2 - a^2 - b^2, c^2 + 2ac + a^2 + b^2, a^2 - b^2 - c^2), \\ R_c &= (b^2 - a^2 - c^2, a^2 - b^2 - c^2, a^2 + 2ab + b^2 + c^2)). \end{aligned}$$

Denote by L_a, L_b and L_c the lines AR_a, BR_b and CR_c, respectively. By using (2), we obtain the equtions of these lines:

$$\begin{aligned} L_a : &\ (a^2 - b^2 + c^2)y + (c^2 - a^2 - b^2)z = 0, \\ L_b : &\ (a^2 - b^2 - c^2)x + (a^2 + b^2 - c^2)z = 0, \\ L_c : &\ (b^2 + c^2 - a^2)x + (b^2 - a^2 - c^2)y = 0. \end{aligned}$$

By using (3), we obtain that the lines L_a, L_b and L_c concur in a point. The problem is solved.

After some additional work, we may identify the point of intersection of these lines, that is, the Prasolov anticevian product of the points. In this problem, the Prasolov anticevian product is the orthocenter of $\triangle ABC$.

Additional problems for the reader:

Prove that the Prasolov anticevian products exist for the pairs of points given in the table below. In the table we also give the Prasolov anticevian products. Recall that the anticomplement of a point P is the image of the homothety with center the centroid and ratio -2. In the table aP denotes the anticomplement of point P.

	U	P	Product
1	Incenter, X(1)	Circumcenter, X(3)	X(84)
2	Circumcenter, X(3)	Nine-point center, X(5)	X(68)
3	Symmedian point, X(6)	Circumcenter, X(3)	X(64)
4	Symmedian point, X(6)	Nine-point center, X(5)	X(4)
5	Centroid, X(2)	Arbitrary point P	aP

References

[1] Douillet, P. (2012). Translation of the Kimberlings Glossary into barycentrics, http://eg-enc.webege.com/htm/links/glossary.pdf

[2] Kimberling, C. Encyclopedia of Triangle Centers, http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[3] Schindler, M. and Cheny, E. (2012). Barycentric Coordinates in Olympiad Geometry. http://www.artofproblemsolving.com/Resources/Papers/Bary_full.pdf