Solition

We will use a powerful computational approach, applicable to a large class of olympiad geometry problems - barycentric coordinates. Denote by a, b and c the side lengths of $\triangle ABC$, $a = BC, b = CA$ and $c = AB$. Recall some formulas (see e.g. [3]). Given a point $U = (u, v, w)$. The vertices of the circumcevian triangle of U are the points

$$\begin{array}{rcl} U_a & = & ((-a^2vw)/(c^2v + b^2w), v, w), \\ U_b & = & (u, (-b^2wu)/(a^2w + c^2u), w), \\ U_c & = & (u, v, (-c^2uv)/(b^2u + a^2v)). \end{array}$$

The normalized barycentric coordinates of U are $(\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w})$. Given point $P = (p, q, r)$. The reflection of point U in point P (U and P must be in normalized barycentric coordinates, see [1], section 3.1) is the point

(1)

$$R = ((p-q-r)u+2p(v+w), (q-p-r)v+2q(u+w), (r-p-q)w+2r(u+v)).$$

The equation of a line L through the points U and P (not necessary in normalized coordinates, see [3], section B1) is

(2) $$L : (vr - wq)x + (wp - ur)y + (uq - vp)z = 0.$$

Three lines $p_ix + q_iy + r_iz = 0,\ i = 1, 2, 3$, are concurrent (see [3], section B1) if and only if

(3) $$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

As in [2], X(n), $n \in \mathbb{N}$, denotes a notable point in the plane of $\triangle ABC$. In this problem $U = (a, b, c)$ is the incenter of $\triangle ABC$, point X(1), and $P = (b+c, c+a, a+b)$ is the Spieker center of $\triangle ABC$, point X(10). Denote by R_a, R_b and R_c the reflections of the vertices of the circumcevian triangle of U in point P, respectively. By using (1), we obtain

$$\begin{array}{rcl} R_a & = & (-a^2 + ab + ac - b^2 - 2cb - c^2, a^2 - ab - c^2 + b^2, a^2 - ac + c^2 - b^2), \\ R_b & = & (-a^2 + ab + c^2 - b^2, a^2 + 2ac - ab + c^2 - cb + b^2, a^2 - c^2 + cb - b^2), \\ R_c & = & (-a^2 + ac - c^2 + b^2, a^2 - c^2 + cb - b^2, a^2 - ac + 2ab + c^2 - cb + b^2). \end{array}$$

Denote by L_a, L_b and L_c the lines AR_a, BR_b and CR_c, respectively. By using (2), we obtain the equations of these lines:

$$\begin{array}{rl} L_a : & (-a^2 + ac - c^2 + b^2)y + (a^2 - ab - c^2 + b^2)z = 0, \\ L_b : & (a^2 - c^2 + cb - b^2)x + (a^2 - ab - c^2 + b^2)z = 0, \\ L_c : & (-a^2 + c^2 - cb + b^2)x + (-a^2 + ac - c^2 + b^2)y = 0. \end{array}$$

By using (3), we obtain that the lines L_a, L_b and L_c concur in a point. The problem is solved.

After some additional work, we may identify the point of intersection of these lines, that is, the Prasolov circumcevian product of the points. In this problem, the Prasolov circumcevian product is the point X(80), that is, the reflection of the incenter of $\triangle ABC$ in the Feuerbach point of $\triangle ABC$.

Additional problems for the reader: Prove that the Prasolov products exist for the pairs of points given in the table below. In the table we also give the Prasolov circumcevian products. If the Prasolov circumcevian product of two points is a point which is not available in [2], we denote it by X(-).

	U	P	product
1	Incenter, X(1)	Nine-Point Center, X(5)	X(4)
2	Incenter, X(1)	Schiffler Point, X(21)	X(3065)
3	Orthocenter, X(4)	Kosnita point, X(54)	X(-)
4	Symmedian point, X(6)	Centroid, X(2)	X(671)
5	Third Power Point, X(32)	Brocard midpoint, X(39)	X(1916)

References

[1] Douillet, P. (2012). Translation of the Kimberlings Glossary into barycentrics, `http://eg-enc.webege.com/htm/links/glossary.pdf`

[2] Kimberling, C. Encyclopedia of Triangle Centers, `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`

[3] Schindler, M. and Cheny, E. (2012). Barycentric Coordinates in Olympiad Geometry. `http://www.artofproblemsolving.com/Resources/Papers/Bary_full.pdf`