Computer-Generated Mathematics: Constructions of the Intangents Triangle

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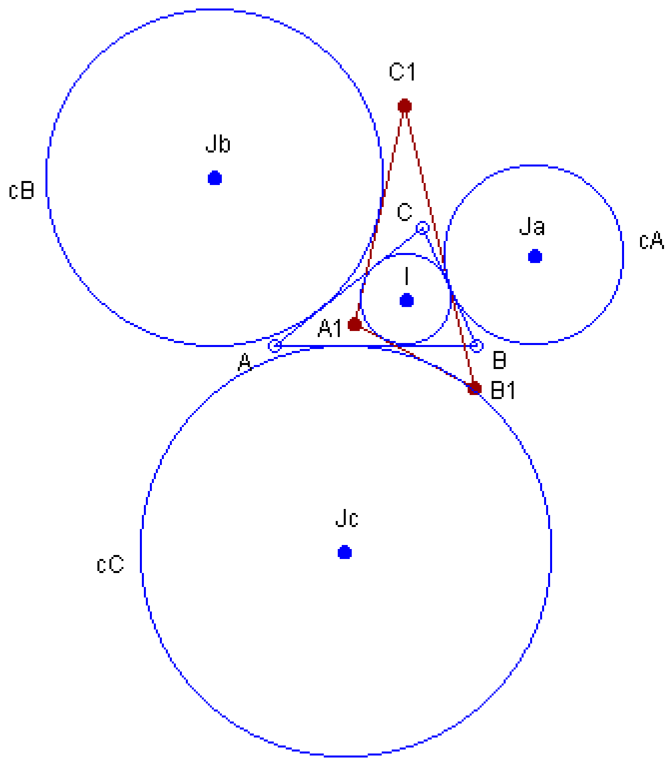
Abstract. By using the computer program "Machine for Questions and Answers", we find 36 different ways to construct the Intangents Triangle by using a straightedge and compass.

Constructions of geometric objects in the plane by using straightedge and compass only, is an essential part of school education in Geometry. In this paper we illustrate the use of the computer program "Machine for Questions and Answers" (The Machine), created by the author of the paper, for discovering of theorems useful for the straightedge-and-compass constructions. We show 36 ways to construct the Intangents Triangle by using straightedge and compass. The method used in this paper could be useful in many similar situations. See e,g. applications of the method for construction of the Apollonius circle [3], Malfatti squares triangle [4], Outer Gallatly-Kiepert triangle [5].

Given a triangle ABC, there are four lines simultaneously tangent to the incircle (with center I) and the A-excircle (with center JA). Of these, three correspond to the sidelines of the triangle, and the fourth is known as the A-intangent The three intangents intersect one another pairwise, and their points of intersection form the *Intangents Triangle*. See [1,2,6,7].

We could construct the Intangents Triangle by using the definition. We have to construct the incircle, the excircles, the intangents and the intersection points of the intangents. See the Figure:



I - Incenter;
Circle centered at I - Incircle;
cA, cB, cC - Excircles;
Ja, Jb, Jc - centers of the Excircles;
The lines BC and B_1C_1 are the internal tangents to the incircle and the A-excircle;
The lines CA and C_1A_1 are the internal tangents to the incircle and the B-excircle;
The lines AB and A_1B_1 are the internal tangents to the incircle and the C-excircle;
$A_1B_1C_1$ - Intangents Triangle.

We use the Machine to find 36 additional ways how to construct the Intangents Triangle. In these ways we do not need to construct the intangents.

We use the following method. We can construct a triangle, if we can construct

- A triangle perspective to the triangle, and the perspector,
- and, a second triangle perspective to the triangle, and the second perspector.

We use the Machine to specify a few theorems given in the paper [6]. We obtain the following theorems (the list below could be extended by the reader by adding additional similar theorems):

Theorem 1. The Intangents Triangle and the Intouch Triangle are perspective with perspector the Incenter.

Theorem 2. The Intangents Triangle and the Incentral Triangle are perspective with perspector the Internal Center of Similitude of the Incircle and the Circumcircle.

Theorem 3. The Intangents Triangle and the Tangential Triangle of the Medial Triangle are homothetic with homothetic center the First Feuerbach Point.

Theorem 4. The Intangents Triangle and the Tangential Triangle of the Euler Triangle are homothetic with homothetic center the Second Feuerbach Point.

Theorem 5. The Intangents Triangle and the Tangential Triangle of the Johnson Triangle are homothetic with homothetic center the Center of the Outer Johnson-Yff Circle.

Theorem 6. The Intangents Triangle and the Tangential Triangle of the Outer Yff Triangle are homothetic with homothetic center the Circumcenter.

Theorem 7. The Intangents Triangle and the Extangents Triangle of the Outer Yff Triangle are homothetic with homothetic center the Moses Point.

Theorem 8. The Intangents Triangle and the Triangle of the reflections of the Incenter in the vertices of the Tangential Triangle are homothetic with homothetic center the Internal Center of Similitude of the Incircle and the Bevan Circle.

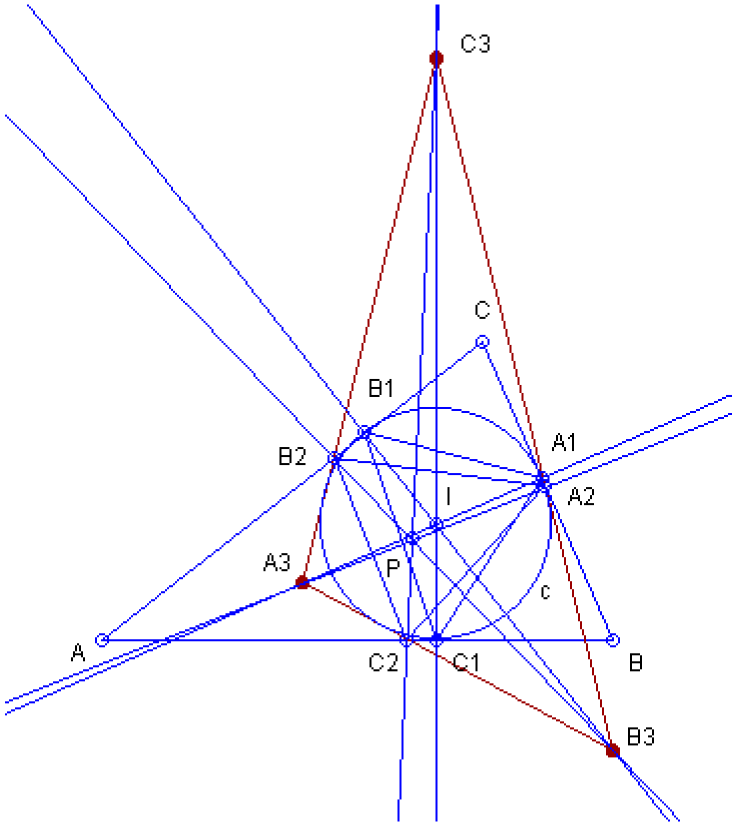
Theorem 9. The Intangents Triangle and the Triangle of the reflections of the vertices of the Tangential Triangle in the Circumcenter are homothetic with homothetic center the External Center of Similitude of the Incircle and the Circumcircle.

The reader may find the definitions in [1,2,7]. The first two theorems are known - see [7, Perspector]. The other theorems are possible new theorems, discovered by the Machine. We invite the reader to prove theorems 3 to 9.

We use the above theorems to obtain 36 ways to construct the Intangents Triangle: The above nine perspectives give 36 ways (Clearly, if the reader extends the list of the perspectives, he will obtain additional ways.)

Solution 1

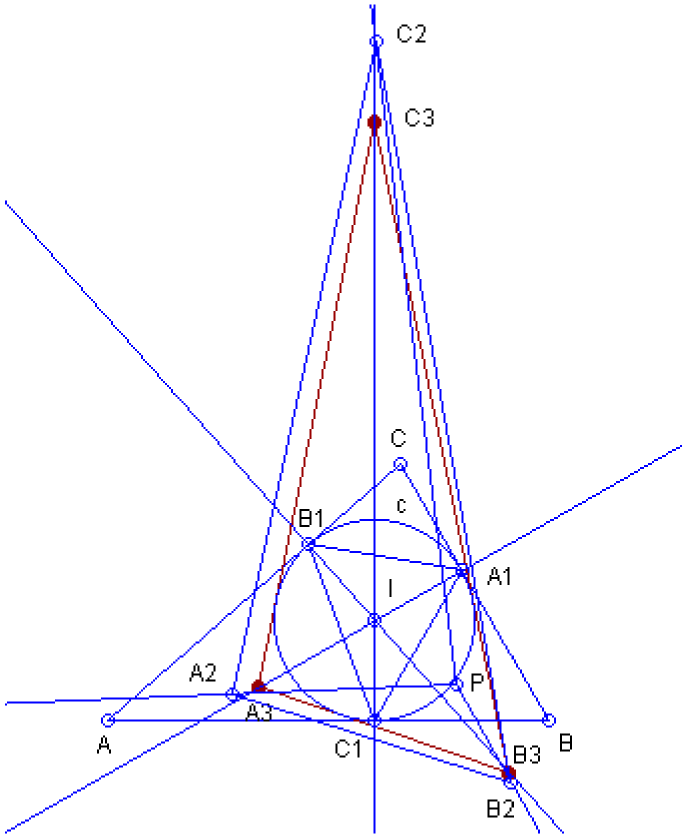
We use Theorems 1 and 2. See the Figure:



c - Incircle;
I - Incenter;
$A_1B_1C_1$ - Intouch Triangle;
P - Internal Center of Similitude of the Incircle and the Circumcircle;
$A_2B_2C_2$ - Incentral Triangle;
A_3 - intersection point of lines IA_1 and PA_2;
B_3 - intersection point of lines IB_1 and PB_2;
C_3 - intersection point of lines IC_1 and PC_2;
$A_3B_3C_3$ - Intangents Triangle.

Solution 2

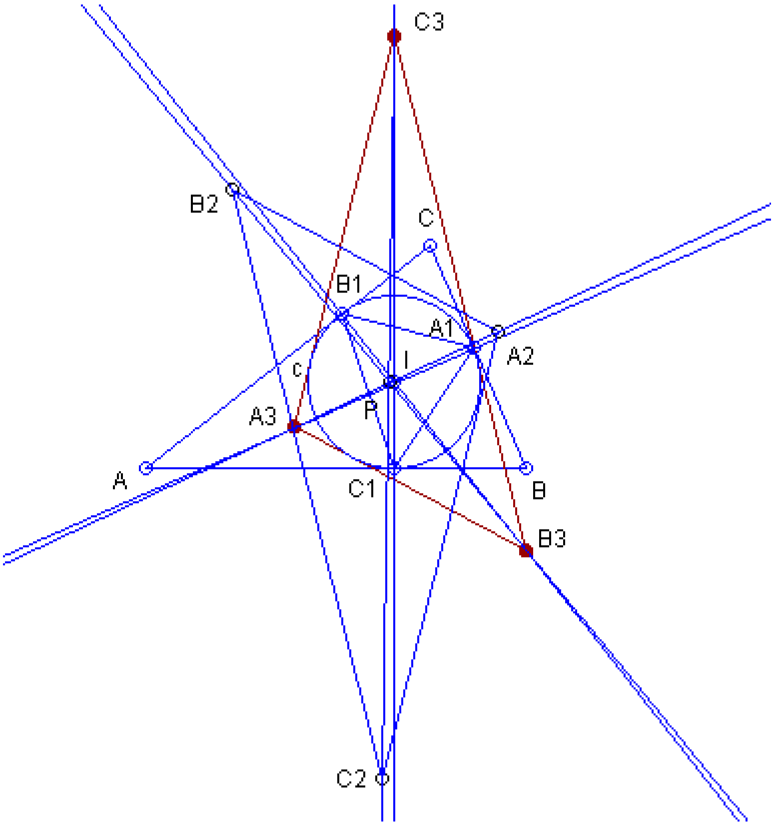
We use Theorems 1 and 3. See the Figure:



c - Incircle;
I - Incenter;
$A_1B_1C_1$ - Intouch Triangle;
P - First Feuerbach Point;
$A_2B_2C_2$ - Tangential Triangle of the Medial Triangle;
A_3 - intersection point of lines IA_1 and PA_2;
B_3 - intersection point of lines IB_1 and PB_2;
C_3 - intersection point of lines IC_1 and PC_2;
$A_3B_3C_3$ - Intangents Triangle.

Solution 3

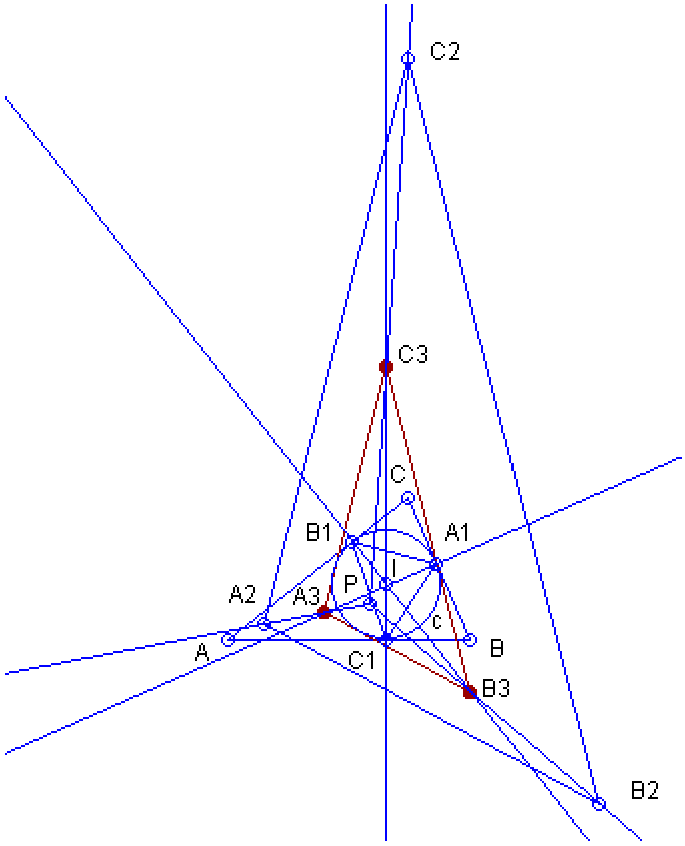
We use Theorems 1 and 4. See the Figure:



c - Incircle;
I - Incenter;
$A_1B_1C_1$ - Intouch Triangle;
P - Second Feuerbach Point;
$A_2B_2C_2$ - Tangential Triangle of the Euler Triangle;
A_3 - intersection point of lines IA_1 and PA_2;
B_3 - intersection point of lines IB_1 and PB_2;
C_3 - intersection point of lines IC_1 and PC_2;
$A_3B_3C_3$ - Intangents Triangle.

Solution 4

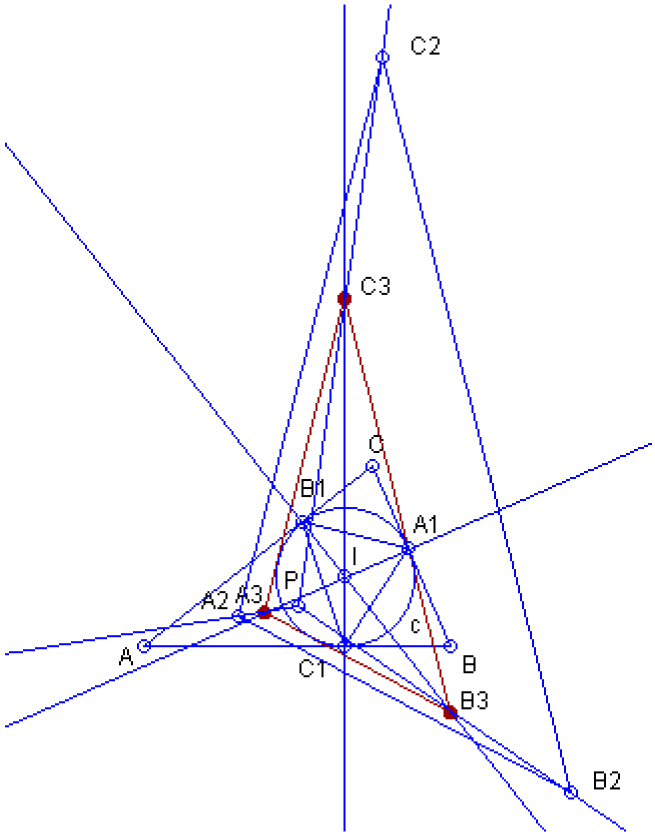
We use Theorems 1 and 5. See the Figure:



c - Incircle;
I - Incenter;
$A_1B_1C_1$ - Intouch Triangle;
P - Center of the Outer Johnson-Yff Circle;
$A_2B_2C_2$ - Tangential Triangle of the Johnson Triangle;
A_3 - intersection point of lines IA_1 and PA_2;
B_3 - intersection point of lines IB_1 and PB_2;
C_3 - intersection point of lines IC_1 and PC_2;
$A_3B_3C_3$ - Intangents Triangle.

Solution 5

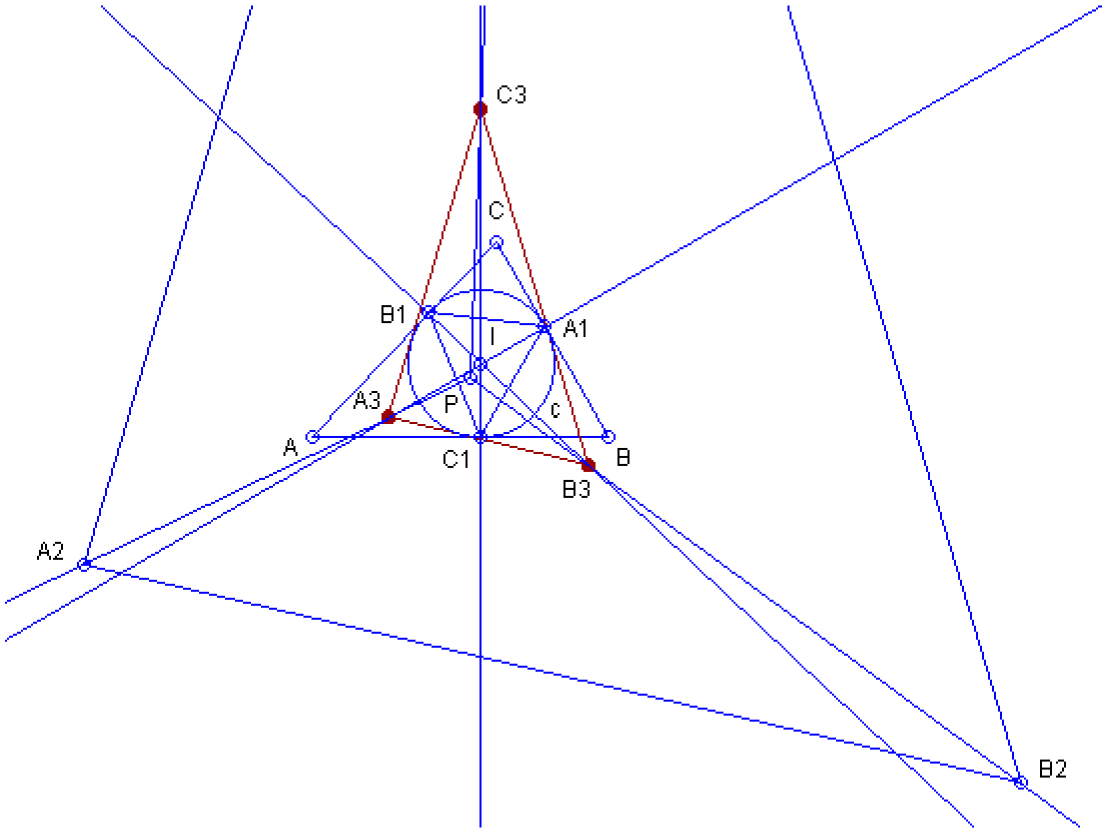
We use Theorems 1 and 6. See the Figure:



c - Incircle;
I - Incenter;
$A_1B_1C_1$ - Intouch Triangle;
P - Circumcenter;
$A_2B_2C_2$ - Tangential Triangle of the Outer Yff Triangle;
A_3 - intersection point of lines IA_1 and PA_2;
B_3 - intersection point of lines IB_1 and PB_2;
C_3 - intersection point of lines IC_1 and PC_2;
$A_3B_3C_3$ - Intangents Triangle.

Solution 6

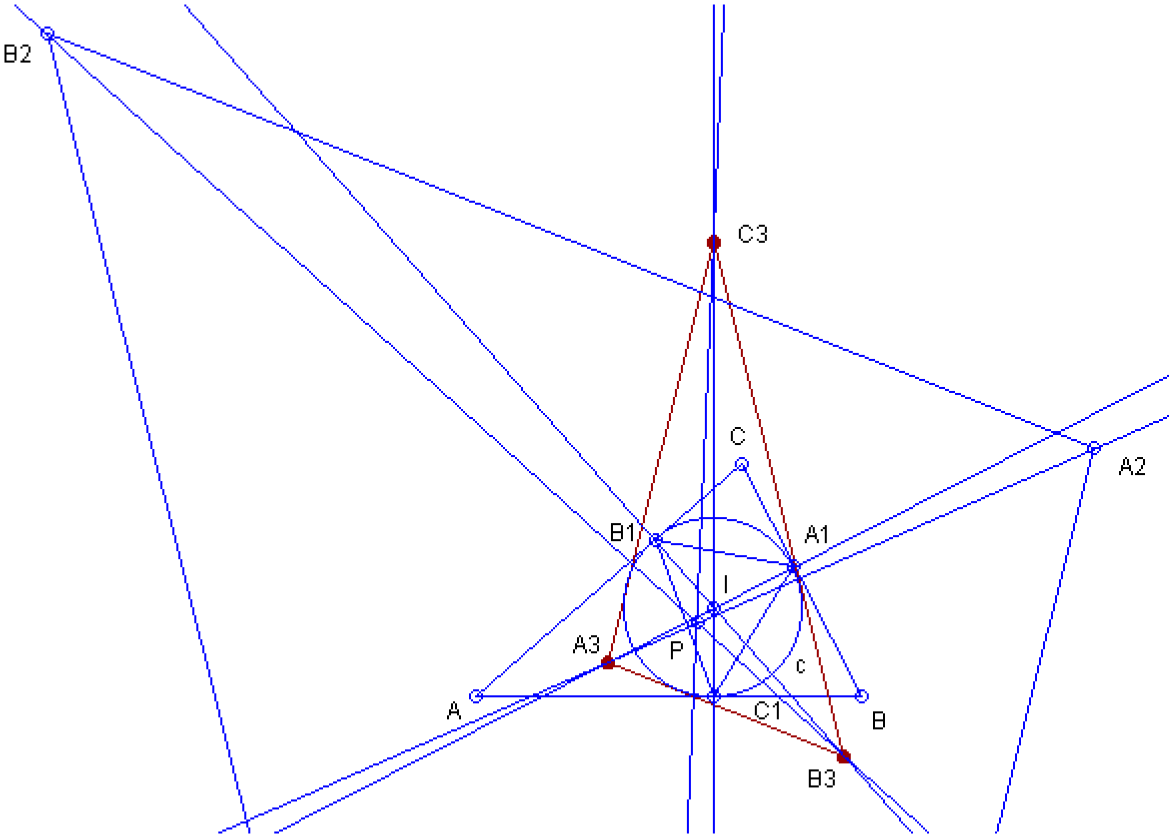
We use Theorems 1 and 7. See the Figure:



c - Incircle;
I - Incenter;
$A_1B_1C_1$ - Intouch Triangle;
P - Moses Point;
$A_2B_2C_2$ - Extangents Triangle of the Outer Yff Triangle;
A_3 - intersection point of lines IA_1 and PA_2;
B_3 - intersection point of lines IB_1 and PB_2;
C_3 - intersection point of lines IC_1 and PC_2 (point C_3 is outside the picture);
$A_3B_3C_3$ - Intangents Triangle.

Solution 7

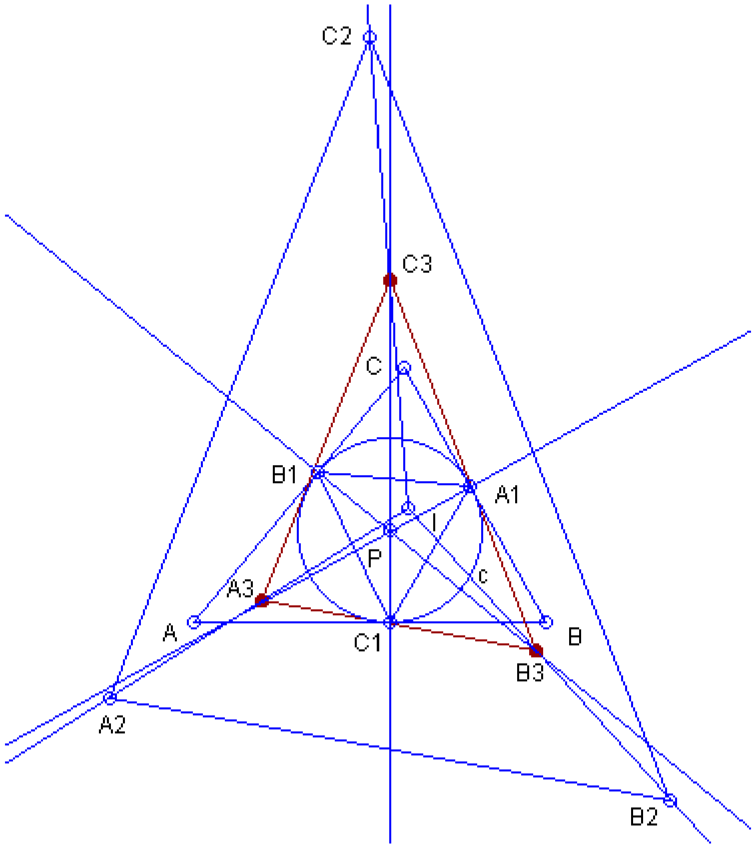
We use Theorems 1 and 8. See the Figure:



c - Incircle;
I - Incenter;
$A_1B_1C_1$ - Intouch Triangle;
P - Internal Center of Similitude of the Incircle and the Bevan Circle;
$A_2B_2C_2$ - Triangle of the reflections of the Incenter in the vertices of the Tangential Triangle;
A_3 - intersection point of lines IA_1 and PA_2;
B_3 - intersection point of lines IB_1 and PB_2;
C_3 - intersection point of lines IC_1 and PC_2 (point C_3 is outside the picture);
$A_3B_3C_3$ - Intangents Triangle.

Solution 8

We use Theorems 1 and 9. See the Figure:



c - Incircle;
I - Incenter;
$A_1B_1C_1$ - Intouch Triangle;
P - External Center of Similitude of the Incircle and the Circumcircle;
$A_2B_2C_2$ - Triangle of the reflections of the vertices of the Tangential Triangle in the Circumcenter;
A_3 - intersection point of lines IA_1 and PA_2;
B_3 - intersection point of lines IB_1 and PB_2;
C_3 - intersection point of lines IC_1 and PC_2;
$A_3B_3C_3$ - Intangents Triangle.

We leave the other solutions to the reader. To obtain the other solutions, we have to use: Theorems 2 and 3, Theorems 2 and 4, etc. It is clear that if we have n perspectives, we obtain n(n-1)/2 different ways how to construct the desired triangle. In this case n = 9, hence we have 36 solutions.

Thanks

The figures in this note are produced by using the program C.a.R. (Compass and Ruler), an amazing program created by Rene Grothmann. The Grothmann's program is available for download in the Web: Rene Grothmann's C.a.R.. It is free and open source. The reader may verify easily the statements of this paper by using C.a.R. Many thanks to Rene Grothmann for his wonderful program.

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