Computer-Generated Mathematics: The Symmedian Point

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Abstract. We illustrate the use of the computer program "Machine for Questions and Answers" (The Machine) for discovering of new theorems in Euclidean Geometry. The paper contains more than 100 new theorems about the Symmedian Point, discovered by the Machine.

Keywords: computer-generated mathematics, Euclidean geometry

"Within ten years a digital computer will discover and prove an important mathematical theorem." (Simon and Newell, 1958).

This is the famous prediction by Simon and Newell [1]. Now is 2008, 50 years later. The first computer program able easily to discover new deep mathematical theorems - The *Machine for Questions and Answers* (The *Machine*) [2,3] has been created by the author of this article, in 2006, that is, 48 years after the prediction. The Machine has discovered a few thousands new mathematical theorems, that is, more than 90% of the new mathematical computer-generated theorems since the prediction by Simon and Newell. In 2006, the Machine has produced the first computer-generated encyclopedia [2].

Given an object (point, triangle, circle, line, etc.), the Machine produces theorems related to the properties of the object. The theorems produced by the Machine are either known theorems, or possible new theorems. A *possible new* theorem means that the theorem is either known theorem, but the source is not available for the author of the Machine, or the theorem is a new theorem. I expect that approximately 75 to 90% of the possible new theorems are new theorems. Although the Machine works completely independent from the human thinking, the theorems produced are surprisingly similar to the theorems produced by the people.

The advantages in using the Machine are as follows. (1) It is not necessary we to be inventive and even it is not necessary we to think. It is enough we to click with the mouse in order to obtain the theorems. (2) The Machine produces complete knowledge. If there exists a theorem related to the object, the Machine discovers the theorem. (3) The people make errors, but the computers do not make errors.

In this paper we illustrate the use of the Machine. We present lists with theorems about the Symmedian point, discovered by the Machine. The lists include a few well known theorems, as well as possible new theorems. The lists contain 152 theorems. I expect that approximately 110 to 120 of these theorems are new theorems. Hence, the paper contains more than 100 new theorems about the Symmedian Point, discovered by the Machine. The reader is invited to select the new theorems and to prove them. There are a few additional lists of theorems about the Symmedian point, produced by the Machine, which are not included in this paper.

The Machine is at a very early stage of development. The current version of the Machine could be considered as a first step to an improved and extended version, able to discover new theorems and to produce encyclopedias in all branches of mathematics.

The reader may find the definitions used in this paper, in [2-4].

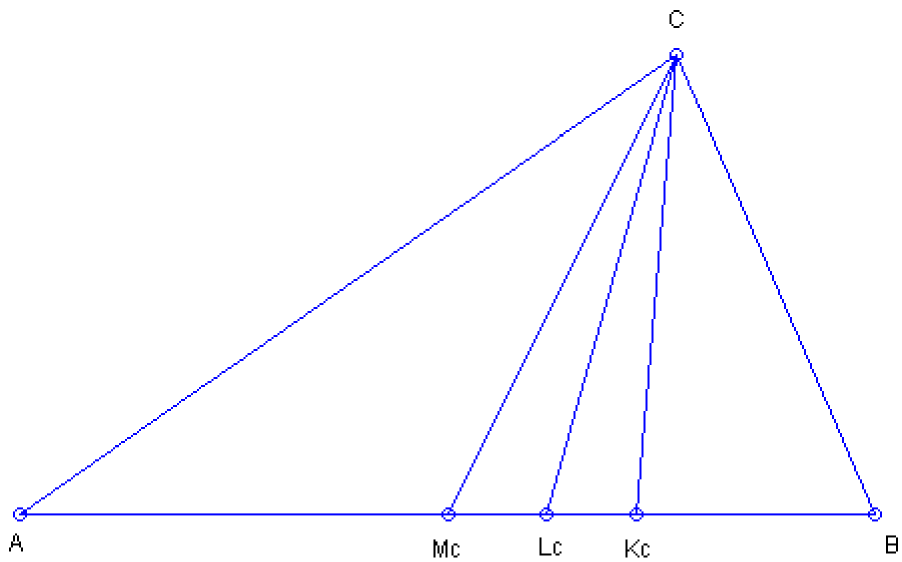
Pedagogical use of the Machine

The Machine could be useful for students and teachers mainly in these directions: (1) The Machine could produce an encyclopedia of Euclidean geometry suitable for school students and teachers. (2) The Machine will give to the school students and teachers the possibility to discover new theorems. (3) The interactive use of the Machine will give to the school students and teachers the possibility to investigate in depth selected problems. (4) The Machine will give to the teachers the possibility easily to produce problems and theorems for textbooks, for use in the classroom, for home works, etc. (5) The Machine will give to the school students and teachers the possibility better to understand the abilities of computers to discover new theorems.

The Symmedian point

Recall the definition of the Symmedian Point.

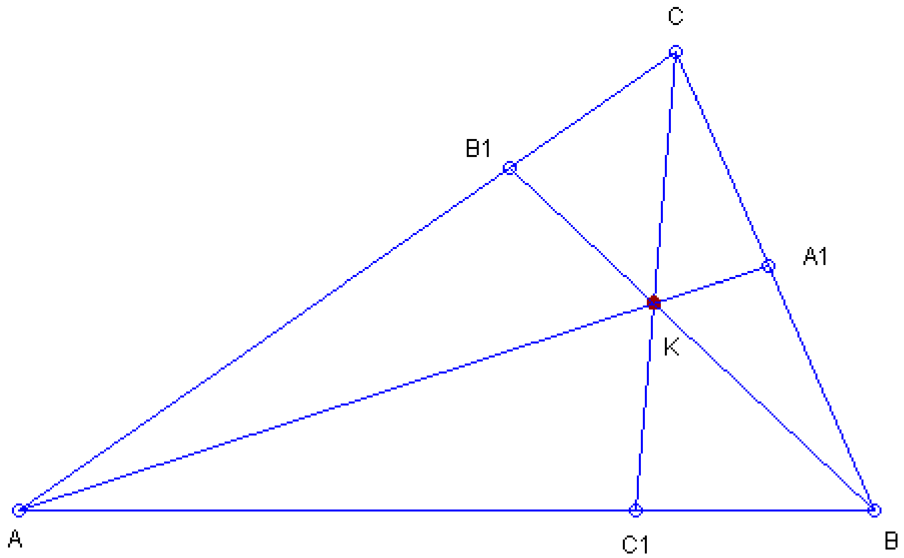
Given a triangle ABC, construct the internal angle bisector CL_c of angle C, and the median CM_c through vertex C. Then construct the cevian CK_c which is the reflection of CM_c in CL_c. The cevian CK_c is called the *symmedian* through vertex C. See the Figure:



CL_c - internal angle bisector of angle C;
CM_c - median through vertex C;

CK_c - symmedian through vertex C;

The three symmedians through vertices A, B, C of a triangle ABC concur at a point. The point of concurrence is called the *Symmedian Point* (sometime the *Lemoine Point*). See the Figure:



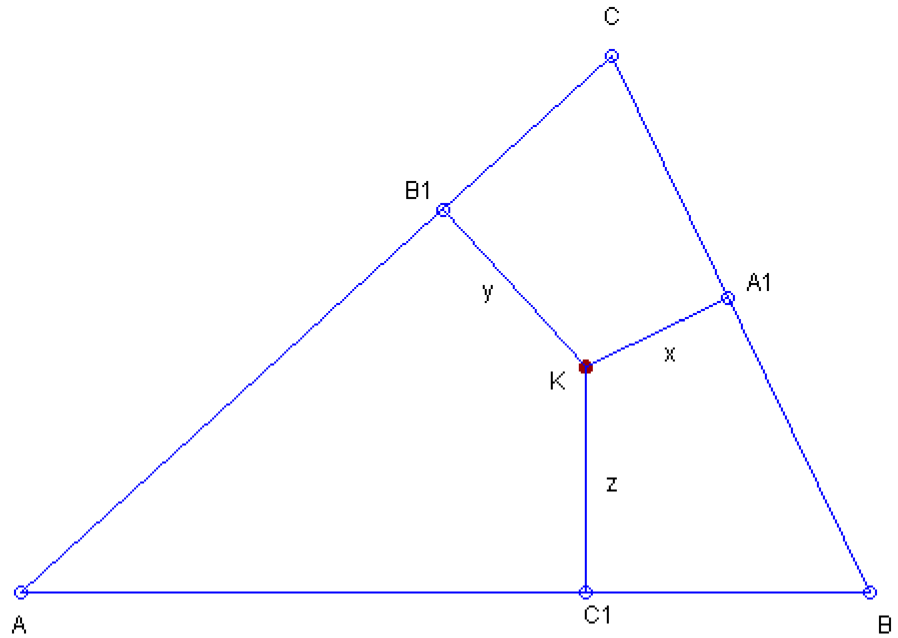
AA_1, BB_1, CC_1 - symmedians through vertices A, B, C, respectively;
K - Symmedian Point = point of concurrence of the three symmedians..

From the Paskalev's book

G. Paskalev in the book [5], sections 27, quoted five famous theorems about the Symmedian Point (called Lemoine point in [5]). Recall these theorems below.

Theorem 1. Given a triangle ABC and a point K in the interior of the triangle. Denote by x, y, z the distances from point K to the sides BC, CA, AB of the triangle. Then the function $f = x^2 + y^2 + z^2$ has minimum iff K is the Symmedian Point.

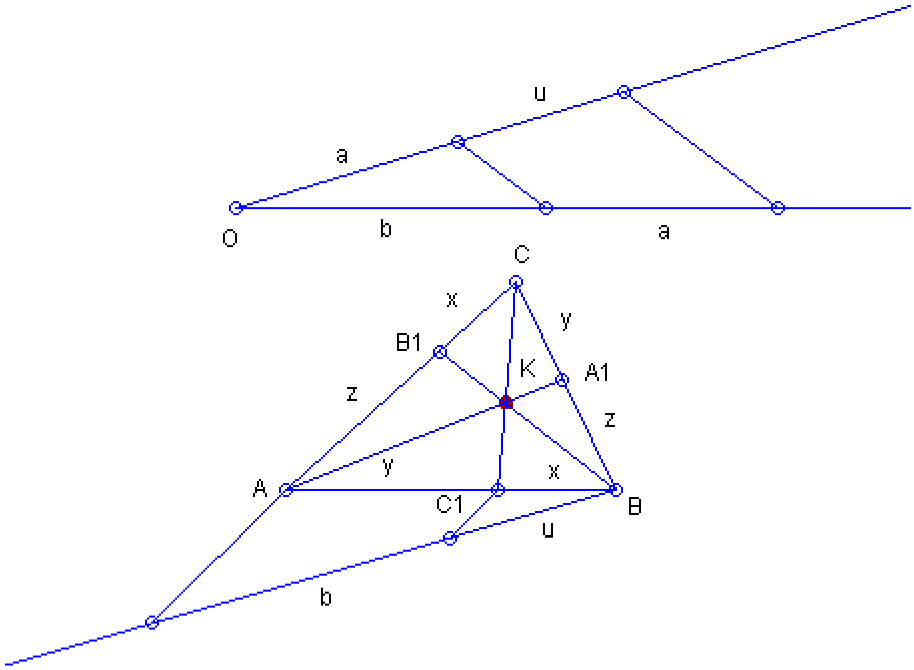
See the Figure:



A_1 - projection of point K on side BC;
B_1 - projection of point K on side CA;
C_1 - projection of point K on side AB;
x - distance from point K to side BC;
y - distance from point K to side CA;
z - distance from point K to side AB;
The function $f = x^2 + y^2 + z^2$ has minimum iff K is the Symmedian Point.

Theorem 2. Given a triangle ABC. Denote by a = BC, b = CA, c = AB the sidelenghts of the triangle, and denote $x = a^2$, $y = b^2$, $z = c^2$. Construct point C_1 dividing internally the directed segment AB (from A to B) in ratio y : x (that is, dividing the directed segment BA in ratio x : y). Construct point A_1 dividing internally the directed segment BC in ratio z : y (that is, dividing the directed segment CB in ratio y : z). Construct point B_1 dividing internally the directed segment CA in ratio x : z (that is, dividing the directed segment AC in ratio z : x). Then lines AA_1, BB_1 and CC_1 concur at a point and the point of concurrence is the Symmedian Point.

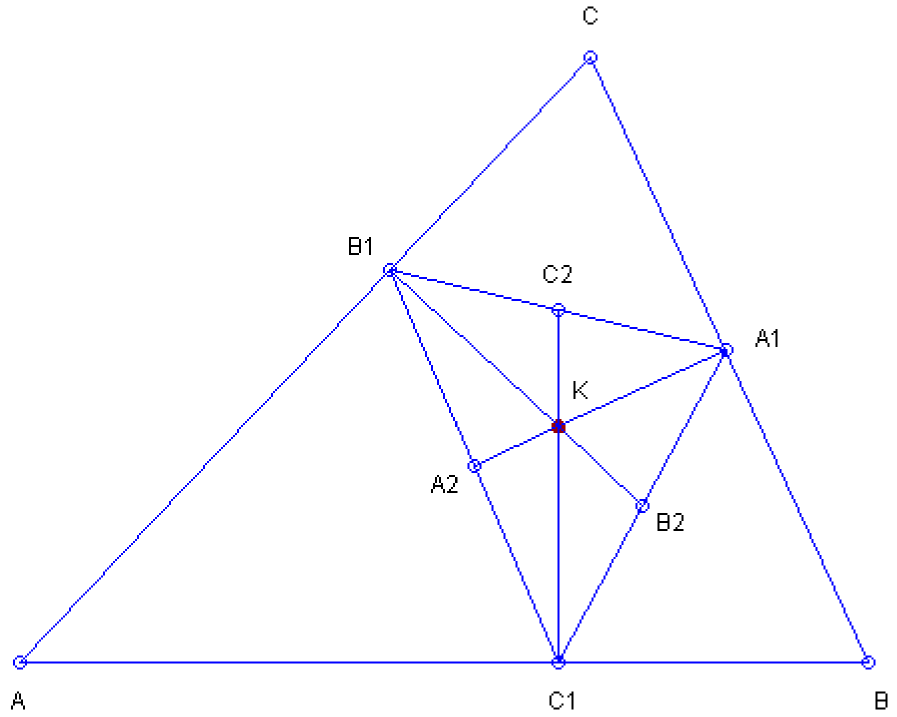
See the Figure:



Construct line segment u = aa/b. Then construct point C_1 dividing line segment BA in ratio x : y = u : b = a^2 : b^2. Lines AA_1, BB_1 and CC_1 concur at the Symmedian Point K.

Theorem 3. The Symmedian Point is the Centroid of the Pedal Triangle of the Symmedian Point.

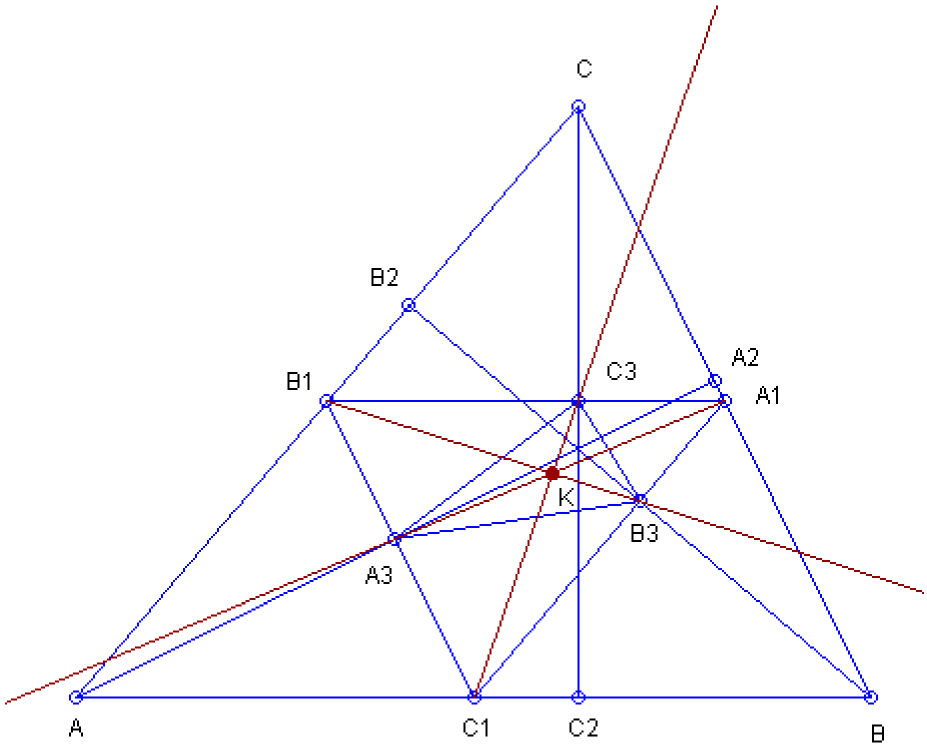
See the Figure:



K - Symmedian Point;
A_1 - orthogonal projection of point K on side BC;
B_1 - orthogonal projection of point K on side CA;
C_1 - orthogonal projection of point K on side AB;
$A_1B_1C_1$ - Pedal Triangle of the Symmedian Point K;
A_2 - midpoint of segment B_1C_1;
B_2 - midpoint of segment C_1A_1;
C_2 - midpoint of segment A_1B_1;
The Symmedian Point is the Centroid of the Pedal Triangle of the Symmedian Point.

Theorem 4. The Symmedian Point is the Perspector of the Medial Triangle and the Half-Altutude Triangle.

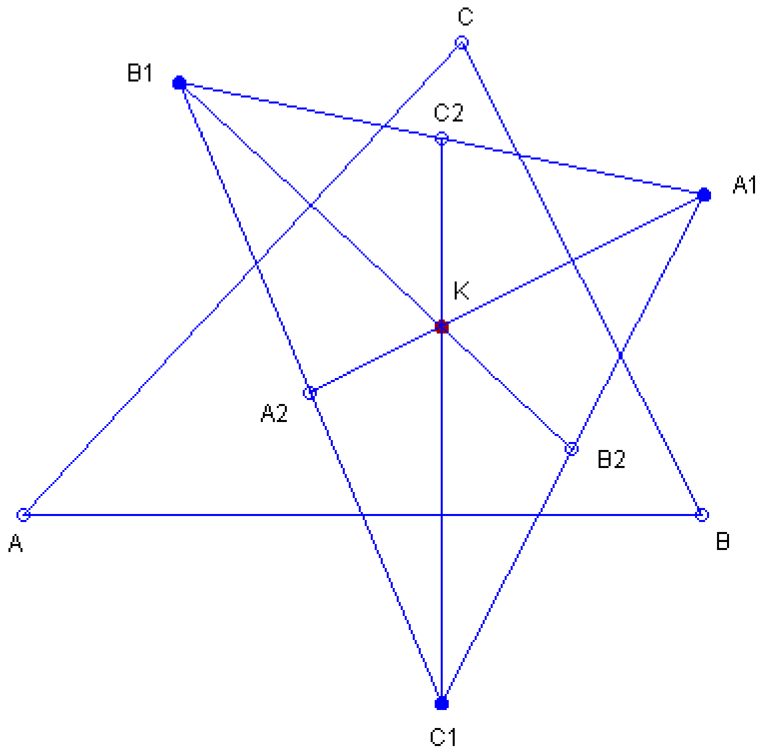
See the Figure:



K - Symmedian Point;
$A_1B_1C_1$ - Medial Triangle;
AA_2, BB_2, CC_2 - altitudes;
A_3 - midpoint of altitude AA_2;
B_3 - midpoint of altitude BB_2;
C_3 - midpoint of altitude CC_2;
$A_3B_3C_3$ - Half-Altutude Triangle;
Lines A_1A_3, B_1B_3 and C_1C_3 concur at a point and the point of concurrence is the Symmedian Point, that is, the Symmedian Point is the Perspector of the Medial Triangle. and the Half-Altutude Triangle.

Theorem 5. The Symmedian Point is the Centroid of the Triangle of reflections of the Symmedian Point in the sides of Triangle ABC.

See the Figure:



K - Symmedian Point;
A_1 - reflection of point K in side BC;
B_1 - reflection of point K in side CA;
C_1 - reflection of point K in side AB;
$A_1B_1C_1$ - Triangle of reflections of the Symmedian Point in the sides of Triangle ABC;
A_2 - midpoint of segment B_1C_1;
B_2 - midpoint of segment C_1A_1;
C_2 - midpoint of segment A_1B_1;
Medians A_1A_2, B_1B_2 and C_1C_2 of triangle $A_1B_1C_1$ concur at the Symmedian point K, that is, the Symmedian Point K is the Centroid of the Triangle of reflections of the Symmedian Point in the sides of Triangle ABC.

In the list of the Roles of the Symmedian Point, given below, are included three of the above theorems as computer-generated theorems: Theorem 3 is no. 1 in the list, theorem 4 is no. 28 in the list, and theorem 5 is no.19 in the list. We comment theorems 1 and 2 in the next two sections.

The Machine and the Extremal Problems

The ability to solve extremal problems is included into the Machine during the first development of the Machine, that is, in April 2006. At this time the Machine discovered a few extremal theorems, including theorem 1 from the Paskalev's book. The ability of the Machine to solve extremal porblems will be considered in another paper.

The Machine and the Defining Segments

Theorem 2 form the Paskalev's book was discovered by the Machine in August 2007 as a part of the general theory of the defining segments. The ability of the Machine to discover theorems about the defining segments will be considered in another paper.

Roles of the Symmedian Point

The list below contains a few theorems about the Roles of the Symmedian Point, discovered by the Machine. Theorems 1, 19 and 28 from the list below are theorems 3, 4 and 5 from the Paskalev's book.

1. The Symmedian Point is the Centroid of the Pedal Triangle of the Symmedian Point.
2. The Symmedian Point is the Symmedian Point of the Circumcevian Triangle of the Schoute Center.
3. The Symmedian Point is the Symmedian Point of the Circumcevian Triangle of the First Beltrami Point.
4. The Symmedian Point is the Symmedian Point of the Circumcevian Triangle of the Second Beltrami Point.
5. The Symmedian Point is the Steiner Point of the Inner Gallatly-Kiepert Triangle.
6. The Symmedian Point is the Second Brocard Point of the First Lemoine-Tucker Triangle.
7. The Symmedian Point is the First Brocard Point of the Second Lemoine-Tucker Triangle.
8. The Symmedian Point is the Circumcenter of the First Cosine-Tucker Triangle.
9. The Symmedian Point is the Circumcenter of the Second Cosine-Tucker Triangle.
10. The Symmedian Point is the Orthocenter of the First Brocard Triangle of the Johnson Triangle.
11. The Symmedian Point is the Symmedian Point of the Fourth Brocard Triangle of the Fourth Brocard Triangle.
12. The Symmedian Point is the Kiepert Center of the Anticomplementary Triangle of the First Brocard Triangle.
13. The Symmedian Point is the Kiepert Center of the First Brocard Triangle of the Anticomplementary Triangle.
14. The Symmedian Point is the Steiner Point of the First Brocard Triangle of the Fourth Brocard Triangle.
15. The Symmedian Point is the Center of the Brocard Circle of the Anticomplementary Triangle of the Johnson Triangle.
16. The Symmedian Point is the de Longchamps Point of the First Brocard Triangle of the Euler Triangle.
17. The Symmedian Point is the Symmedian Point of the Euler Triangle of the Symmedian Point.
18. The Symmedian Point is the Circumcenter of the Triangle of the reflections of the Centroid in the sides of Triangle ABC.
19. The Symmedian Point is the Centroid of the Triangle of the reflections of the Symmedian Point in the sides of Triangle ABC.
20. The Symmedian Point is the Symmedian Point of the Triangle of the reflections of the Schoute Center in the sides of Triangle ABC.
21. The Symmedian Point is the Center of the Brocard Circle of the Triangle of the reflections of the First Brocard Point in the sides of Triangle ABC.
22. The Symmedian Point is the Center of the Brocard Circle of the Triangle of the reflections of the Second Brocard Point in the sides of Triangle ABC.

23. The Symmedian Point is the Centroid of the Hatzipolakis Triangle of the Symmedian Point.
24. The Symmedian Point is the Center of the Brocard Circle of the Anticomplementary Triangle of the Anticomplementary Triangle of the Euler Triangle.
25. The Symmedian Point is the Center of the Second Droz-Farny Circle of the First Cosine-Tucker Triangle.
26. The Symmedian Point is the Inverse of the Schoute Center in the Moses Circle.
27. The Symmedian Point is the External Center of Similitude of the Circumcircle and the Half-Moses Circle.
28. The Symmedian Point is the Perspector of the Medial Triangle and the Half-Altutude Triangle.
29. The Symmedian Point is the Perspector of the Symmedial Triangle and the Second Brocard Triangle.
30. The Symmedian Point is the Perspector of the Tangential Triangle and the Second Brocard Triangle.
31. The Symmedian Point is the Homothetic Center of the Pedal Triangle of the Third Power Point and the Neuberg Triangle.
32. The Symmedian Point is the Homothetic Center of the Pedal Triangle of the Brocard Midpoint and the Reflected Neuberg Triangle.
33. The Symmedian Point is the Homothetic Center of the Pedal Triangle of the Third Power Point and the Inner Lemoine-Kiepert Triangle.
34. The Symmedian Point is the Homothetic Center of the Pedal Triangle of the Brocard Midpoint and the Outer Lemoine-Kiepert Triangle.
35. The Symmedian Point is the Homothetic Center of the Pedal Triangle of the Center of the Brocard Circle and the Outer Gallatly-Kiepert Triangle.
36. The Symmedian Point is the Homothetic Center of the Pedal Triangle of the Inner Kenmotu Point and the Outer Vecten Triangle.
37. The Symmedian Point is the Homothetic Center of the Pedal Triangle of the Outer Kenmotu Point and the Inner Vecten Triangle.
38. The Symmedian Point is the Homothetic Center of the Pedal Triangle of the Danneels-Apollonius Prespector and the First Apollonius-Kiepert Triangle.
39. The Symmedian Point is the Perspector of the Antipedal Triangle of the Circumcenter and the Circumcevian Triangle of the Symmedian Point.
40. The Symmedian Point is the Perspector of the Antipedal Triangle of the Circumcenter and the Second Brocard Triangle.
41. The Symmedian Point is the Perspector of the Circumcevian Triangle of the Symmedian Point and the Second Brocard Triangle.
42. The Symmedian Point is the Homothetic Center of the Orthic Triangle and the Anticomplementary Triangle of the Tangential Triangle.
43. The Symmedian Point is the Perspector of the Symmedial Triangle and the Medial Triangle of the Orthic Triangle.
44. The Symmedian Point is the Homothetic Center of the Tangential Triangle and the Medial Triangle of the Orthic Triangle.
45. The Symmedian Point is the Perspector of the Second Brocard Triangle and the Medial Triangle of the Orthic Triangle.
46. The Symmedian Point is the Perspector of the Fourth Brocard Triangle and the Symmedial Triangle of the Fourth Brocard Triangle.
47. The Symmedian Point is the Perspector of the Fourth Brocard Triangle and the Tangential Triangle of the Fourth Brocard Triangle.
48. The Symmedian Point is the Perspector of the Fourth Brocard Triangle and the

Second Brocard Triangle of the Fourth Brocard Triangle.

49. The Symmedian Point is the Perspector of the Outer Grebe Triangle and the Medial Triangle of the Orthic Triangle.
50. The Symmedian Point is the Perspector of the Inner Grebe Triangle and the Medial Triangle of the Orthic Triangle.
51. The Symmedian Point is the Perspector of Triangle ABC and the Stevanovic Triangle of the Symmedian Points of the Triangulation triangles of the First Isodynamic Point.
52. The Symmedian Point is the Perspector of Triangle ABC and the Stevanovic Triangle of the Symmedian Points of the Triangulation triangles of the Second Isodynamic Point.
53. The Symmedian Point is the Homothetic Center of Triangle ABC and the Triangle of the Symmedian Points of the Corner Triangles of the Centroid.
54. The Symmedian Point is the Perspector of Triangle ABC and the Triangle of the Centroids of the Corner Triangles of the Orthocenter.
55. The Symmedian Point is the Perspector of Triangle ABC and the Triangle of the reflections of the Centroid in the sides of the Excentral Triangle.
56. The Symmedian Point is the Homothetic Center of the Incentral Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Incentral Triangle.
57. The Symmedian Point is the Perspector of the Incentral Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Anticevian Triangle of the Internal Center of Similitude of the Incircle and the Circumcircle.
58. The Symmedian Point is the Perspector of the Medial Triangle and the Triangle of the Symmedian Points of the Triangulation Triangles of the Centroid.
59. The Symmedian Point is the Homothetic Center of the Medial Triangle and the Triangle of the Centroids of the Triangulation Triangles of the Symmedian Point.
60. The Symmedian Point is the Homothetic Center of the Medial Triangle and the Triangle of the Symmedian Points of the Anticevian Corner Triangles of the Centroid.
61. The Symmedian Point is the Perspector of the Medial Triangle and the Triangle of the Centroids of the Anticevian Corner Triangles of the Circumcenter.
62. The Symmedian Point is the Homothetic Center of the Medial Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Medial Triangle.
63. The Symmedian Point is the Perspector of the Medial Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Anticevian Triangle of the Circumcenter.
64. The Symmedian Point is the Perspector of the Orthic Triangle and the Triangle of the Orthocenters of the Triangulation Triangles of the Circumcenter.
65. The Symmedian Point is the Homothetic Center of the Orthic Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Orthic Triangle.
66. The Symmedian Point is the Perspector of the Symmedial Triangle and the Triangle of the Symmedian Points of the Corner Triangles of the Centroid.
67. The Symmedian Point is the Perspector of the Symmedial Triangle and the Triangle of the Centroids of the Corner Triangles of the Orthocenter.
68. The Symmedian Point is the Perspector of the Symmedial Triangle and the Triangle of the reflections of the Centroid in the sides of the Excentral Triangle.
69. The Symmedian Point is the Homothetic Center of the Symmedial Triangle and the

Triangle of the reflections of the Symmedian Point in the vertices of the Symmedial Triangle.
70. The Symmedian Point is the Perspector of the Symmedial Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Tangential Triangle.
71. The Symmedian Point is the Homothetic Center of the Intouch Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Intouch Triangle.
72. The Symmedian Point is the Homothetic Center of the Extouch Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Extouch Triangle.
73. The Symmedian Point is the Homothetic Center of the Excentral Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Excentral Triangle.
74. The Symmedian Point is the Homothetic Center of the Anticomplementary Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Anticomplementary Triangle.
75. The Symmedian Point is the Perspector of the Tangential Triangle and the Stevanovic Triangle of the Symmedian Points of the Triangulation triangles of the First Isodynamic Point.
76. The Symmedian Point is the Perspector of the Tangential Triangle and the Stevanovic Triangle of the Symmedian Points of the Triangulation triangles of the Second Isodynamic Point.
77. The Symmedian Point is the Perspector of the Tangential Triangle and the Triangle of the Symmedian Points of the Corner Triangles of the Centroid.
78. The Symmedian Point is the Perspector of the Tangential Triangle and the Triangle of the Centroids of the Corner Triangles of the Orthocenter.
79. The Symmedian Point is the Perspector of the Tangential Triangle and the Triangle of the reflections of the Centroid in the sides of the Excentral Triangle.
80. The Symmedian Point is the Perspector of the Tangential Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Symmedial Triangle.
81. The Symmedian Point is the Homothetic Center of the Tangential Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Tangential Triangle.
82. The Symmedian Point is the Perspector of the Half-Altitude Triangle and the Triangle of the Symmedian Points of the Triangulation Triangles of the Centroid.
83. The Symmedian Point is the Perspector of the Half-Altitude Triangle and the Triangle of the First Feuerbach Points of the Triangulation Triangles of the Circumcenter.
84. The Symmedian Point is the Perspector of the Half-Altitude Triangle and the Triangle of the Kiepert Centers of the Triangulation Triangles of the Circumcenter.
85. The Symmedian Point is the Perspector of the Half-Altitude Triangle and the Triangle of the Centroids of the Triangulation Triangles of the Symmedian Point.
86. The Symmedian Point is the Perspector of the Half-Altitude Triangle and the Triangle of the Symmedian Points of the Anticevian Corner Triangles of the Centroid.
87. The Symmedian Point is the Perspector of the Half-Altitude Triangle and the Triangle of the Centroids of the Anticevian Corner Triangles of the Circumcenter.
88. The Symmedian Point is the Perspector of the Half-Altitude Triangle and the Triangle of the First Feuerbach Points of the Anticevian Corner Triangles of the Symmedian Point.

89. The Symmedian Point is the Perspector of the Half-Altitude Triangle and the Triangle of the Kiepert Centers of the Anticevian Corner Triangles of the Symmedian Point.
90. The Symmedian Point is the Perspector of the Half-Altitude Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Medial Triangle.
91. The Symmedian Point is the Homothetic Center of the Half-Altitude Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Anticevian Triangle of the Circumcenter.
92. The Symmedian Point is the Perspector of the Reflection Triangle and the Triangle of the Orthocenters of the Triangulation Triangles of the Kosnita Point.
93. The Symmedian Point is the Perspector of the Second Brocard Triangle and the Stevanovic Triangle of the Symmedian Points of the Triangulation triangles of the First Isodynamic Point.
94. The Symmedian Point is the Perspector of the Second Brocard Triangle and the Stevanovic Triangle of the Symmedian Points of the Triangulation triangles of the Second Isodynamic Point.
95. The Symmedian Point is the Perspector of the Second Brocard Triangle and the Triangle of the Symmedian Points of the Corner Triangles of the Centroid.
96. The Symmedian Point is the Perspector of the Second Brocard Triangle and the Triangle of the Centroids of the Corner Triangles of the Orthocenter.
97. The Symmedian Point is the Perspector of the Second Brocard Triangle and the Triangle of the reflections of the Centroid in the sides of the Excentral Triangle.
98. The Symmedian Point is the Perspector of the Second Brocard Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Symmedial Triangle.
99. The Symmedian Point is the Perspector of the Second Brocard Triangle and the Triangle of the reflections of the Symmedian Point in the vertices of the Tangential Triangle.
100. The Symmedian Point is the Perspector of the Fourth Brocard Triangle and the Stevanovic Triangle of the Inner Napoleon Points of the Triangulation triangles of the First Isodynamic Point.
101. The Symmedian Point is the Perspector of the Lucas Central Triangle and the Triangle of the Outer Kenmotu Points of the Triangulation Triangles of the First Isodynamic Point.
102. The Symmedian Point is the Homothetic Center of the Neuberg Triangle and the Stevanovic Triangle of the Orthocenters of the Triangulation triangles of the Third Power Point.
103. The Symmedian Point is the Homothetic Center of the Reflected Neuberg Triangle and the Stevanovic Triangle of the Orthocenters of the Triangulation triangles of the Brocard Midpoint.
104. The Symmedian Point is the Perspector of the Medial Triangle and the Orthic Triangle of the Medial Triangle of the Medial Triangle.
105. The Symmedian Point is the Product of the Circumcenter and the Orthocenter.
106. The Symmedian Point is the Product of the Gergonne Point and the Internal Center of Similitude of the Incircle and the Circumcircle.
107. The Symmedian Point is the Product of the Inner Fermat Point and the Second Isodynamic Point.
108. The Symmedian Point is the Product of the First Isodynamic Point and the

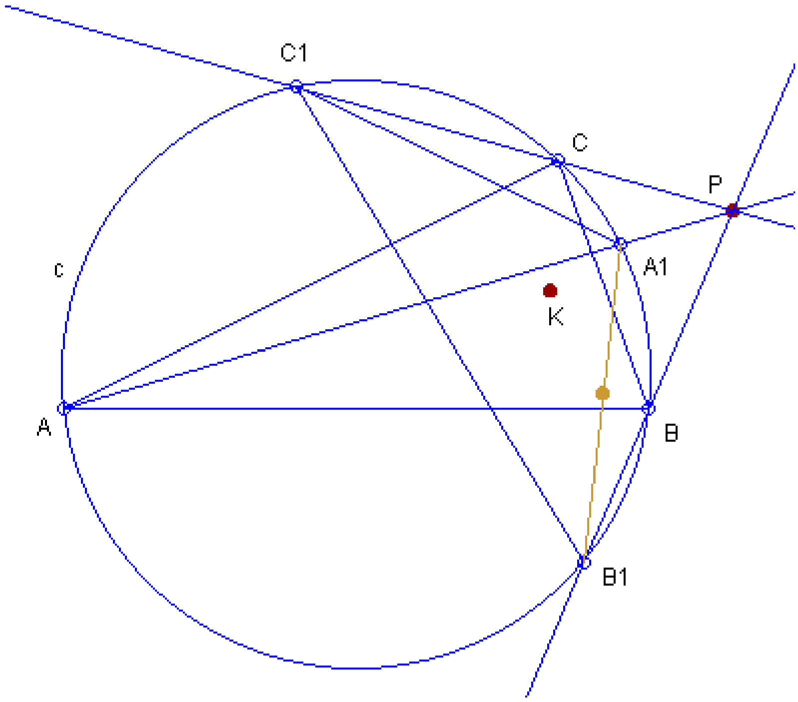
Outer Fermat Point.

109. The Symmedian Point is the Product of the Gibert Point and the Prasolov Point.
110. The Symmedian Point is the Product of the Grinberg Point and the Isogonal Conjugate of the Grinberg Point.
111. The Symmedian Point is the Product of the Kosnita Point and the Nine-Point Center.
112. The Symmedian Point is the Product of the External Center of Similitude of the Incircle and the Circumcircle and the Nagel Point.
113. The Symmedian Point is the Product of the Inner Vecten Point and the Outer Kenmotu Point.
114. The Symmedian Point is the Product of the Inner Kenmotu Point and the Outer Vecten Point.
115. The Symmedian Point is the Product of the First Brocard Point and the Second Brocard Point.
116. The Symmedian Point is the Product of the Mittenpunkt and the Isogonal Conjugate of the Mittenpunkt.
117. The Symmedian Point is the Product of the Spieker Center and the Isogonal Conjugate of the Spieker Center.
118. The Symmedian Point is the Product of the Schiffler Point and the Orthocenter of the Intouch Triangle.
119. The Symmedian Point is the Product of the Second Power Point and the Isotomic Conjugate of the Incenter.
120. The Symmedian Point is the Product of the Third Power Point and the Isotomic Conjugate of the Symmedian Point.
121. The Symmedian Point is the Product of the Congruent Isoscelizers Point and the Perspector of Triangle ABC and the Extouch Triangle of the Intouch Triangle.
122. The Symmedian Point is the Product of the Centroid of the Orthic Triangle and the Isotomic Conjugate of the Nine-Point Center.
123. The Symmedian Point is the Product of the Symmedian Point of the Anticomplementary Triangle and the Homothetic Center of the Orthic Triangle and the Tangential Triangle.

We illustrate a few of the above theorems. We invite the reader to select the new theorems and to prove them.

Theorem 2. The Symmedian Point is the Symmedian Point of the Circumcevian Triangle of the Schoute Center.

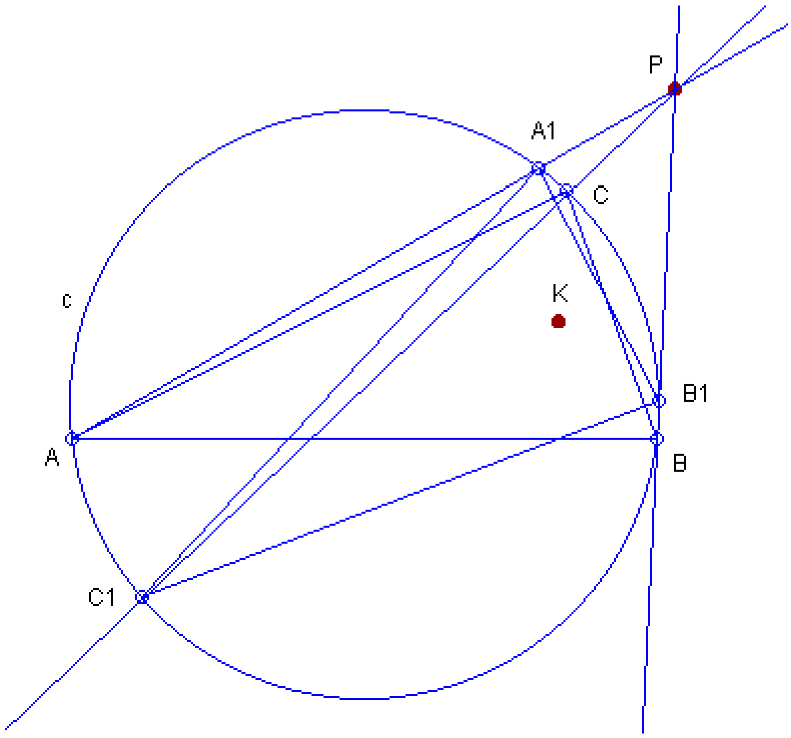
See the Figure:



K - Symmedian Point;
c - circumcircle;
P - Schoute Center
A_1 - intersection point of line AP and circle c (other than point A);
B_1 - intersection point of line BP and circle c (other than point B);
C_1 - intersection point of line CP and circle c (other than point C);
$A_1B_1C_1$ - Circumcevian Triangle of the Schoute Center;
K is the Symmedian point of triangle $A_1B_1C_1$, that is, the Symmedian Point is the Symmedian Point of the Circumcevian Triangle of the Schoute Center.

Theorem 3. The Symmedian Point is the Symmedian Point of the Circumcevian Triangle of the First Beltrami Point.

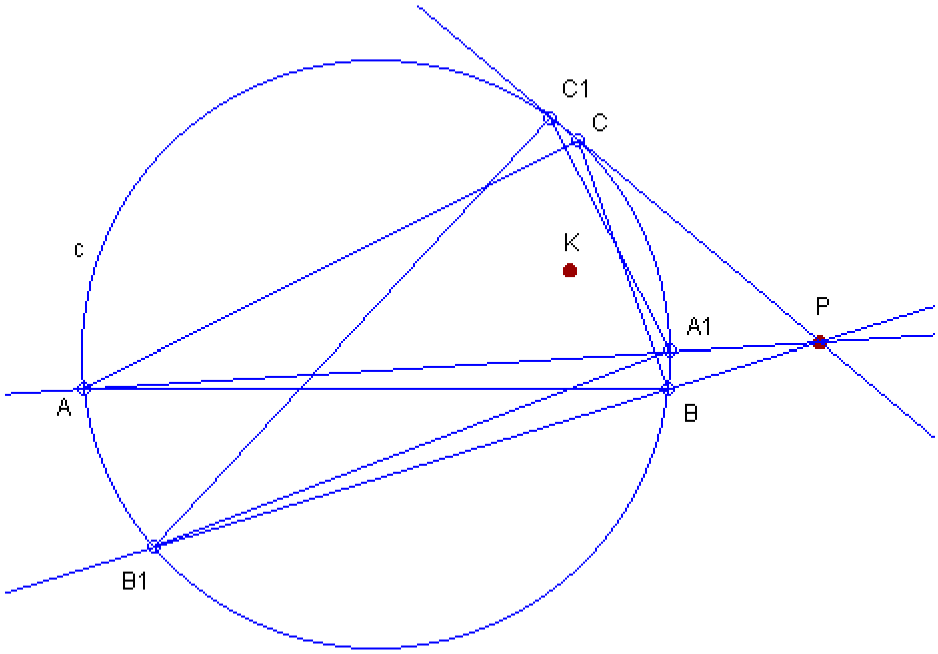
See the Figure:



K - Symmedian Point;
c - circumcircle;
P - First Beltrami Point
A_1 - intersection point of line AP and circle c (other than point A);
B_1 - intersection point of line BP and circle c (other than point B);
C_1 - intersection point of line CP and circle c (other than point C);
$A_1B_1C_1$ - Circumcevian Triangle of the First Beltrami Point;
K is the Symmedian point of triangle $A_1B_1C_1$, that is, the Symmedian Point is the Symmedian Point of the Circumcevian Triangle of the First Beltrami Point.

Theorem 4. The Symmedian Point is the Symmedian Point of the Circumcevian Triangle of the Second Beltrami Point.

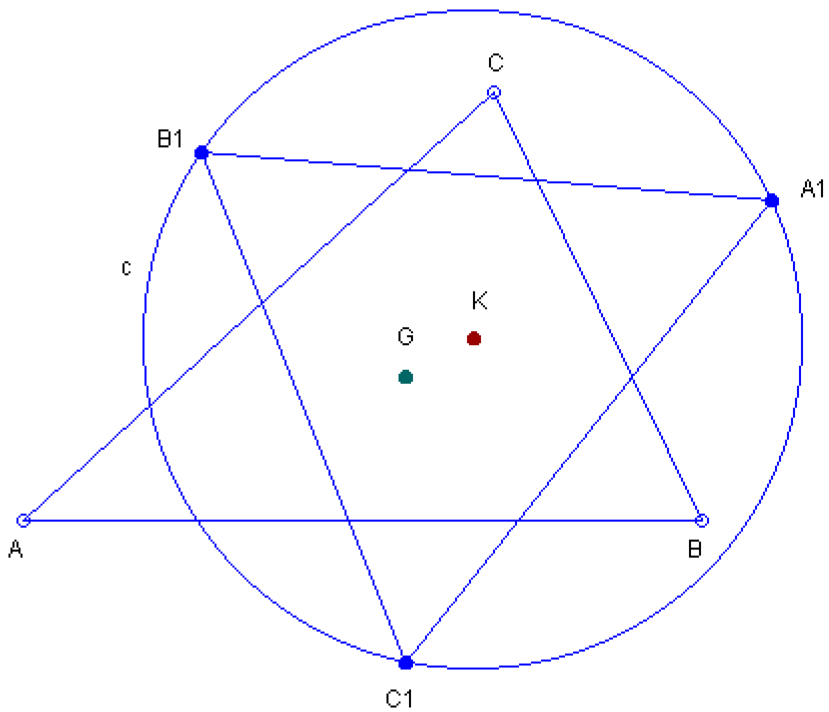
See the Figure:



K - Symmedian Point;
c - circumcircle;
P - Second Beltrami Point
A_1 - intersection point of line AP and circle c (other than point A);
B_1 - intersection point of line BP and circle c (other than point B);
C_1 - intersection point of line CP and circle c (other than point C);
$A_1B_1C_1$ - Circumcevian Triangle of the Second Beltrami Point;
K is the Symmedian point of triangle $A_1B_1C_1$, that is, the Symmedian Point is the Symmedian Point of the Circumcevian Triangle of the Second Beltrami Point.

Theorem 18. The Symmedian Point is the Circumcenter of the Triangle of reflections of the Centroid in the sides of Triangle ABC.

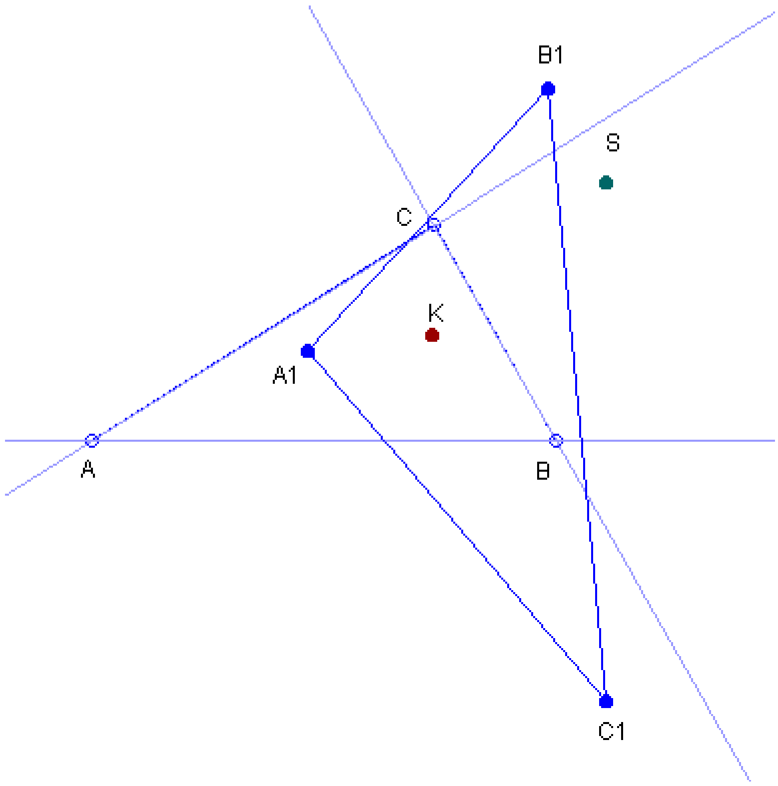
See the Figure:



K - Symmedian Point;
G - Centroid;
A_1 - reflection of point G in side BC;
B_1 - reflection of point G in side CA;
C_1 - reflection of point G in side AB;
$A_1B_1C_1$ - Triangle of reflections of the Centroid in the sides of Triangle ABC;
c - Circumcircle of triangle $A_1B_1C_1$;
K is the Circumcenter of triangle $A_1B_1C_1$, that is, the Symmedian Point is the Circumcenter of the Triangle of reflections of the Centroid in the sides of Triangle ABC.

Theorem 20. The Symmedian Point is the Symmedian Point of the Triangle of reflections of the Schoute Center in the sides of Triangle ABC.

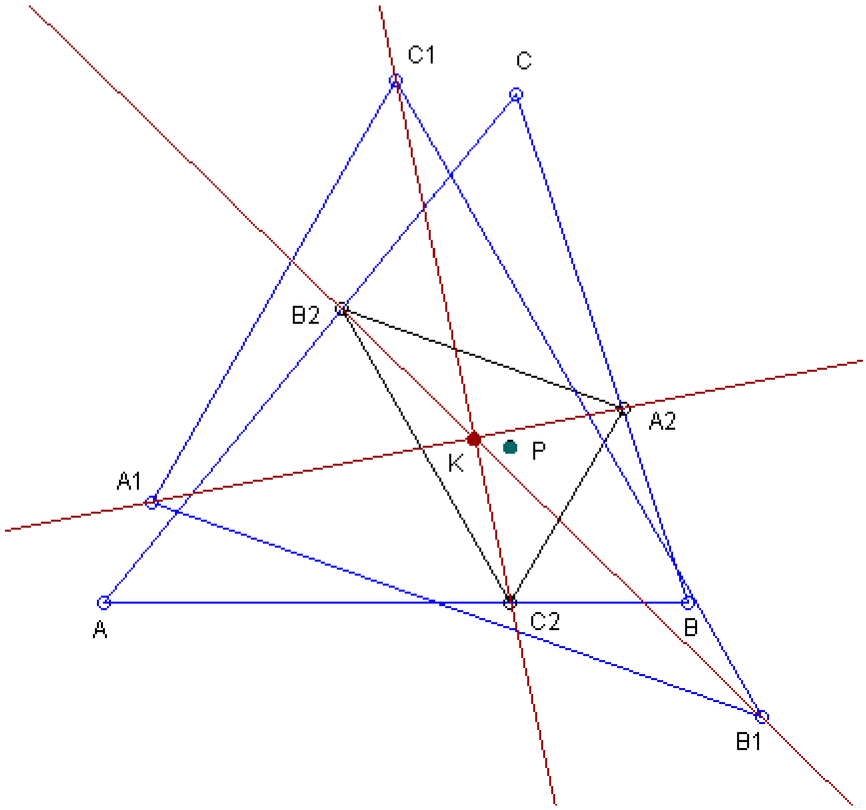
See the Figure:



K - Symmedian Point;
S - Schoute Center;
A_1 - reflection of point S in sideline BC;
B_1 - reflection of point S in sideline CA;
C_1 - reflection of point S in sideline AB;
$A_1B_1C_1$ - Triangle of reflections of the Schoute Center in the sides of Triangle ABC;
K is the Symmedian Point of triangle $A_1B_1C_1$, that is, the Symmedian Point is the Symmedian Point of the Triangle of reflections of the Schoute Center in the sides of Triangle ABC.

Theorem 31. The Symmedian Point is the Homothetic Center of the Pedal Triangle of the Third Power Point and the Neuberg Triangle.

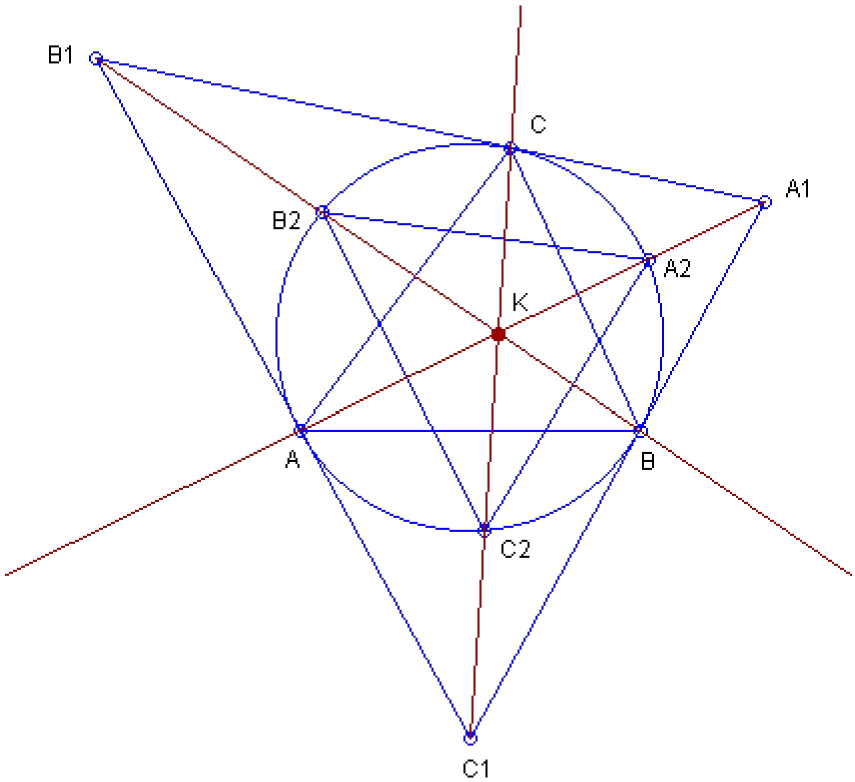
See the Figure:



K - Symmedian Point;
$A_1B_1C_1$ - Neuberg Triangle;
P - Third Power Point;
$A_2B_2C_2$ - Pedal Triangle of the Third Power Point;
K is the Homothetic Center of triangles $A_1B_1C_1$ and $A_2B_2C_2$, that is, the Symmedian Point is the Homothetic Center of the Pedal Triangle of the Third Power Point and the Neuberg Triangle.

Theorem 39. The Symmedian Point is the Perspector of the Antipedal Triangle of the Circumcenter and the Circumcevian Triangle of the Symmedian Point.

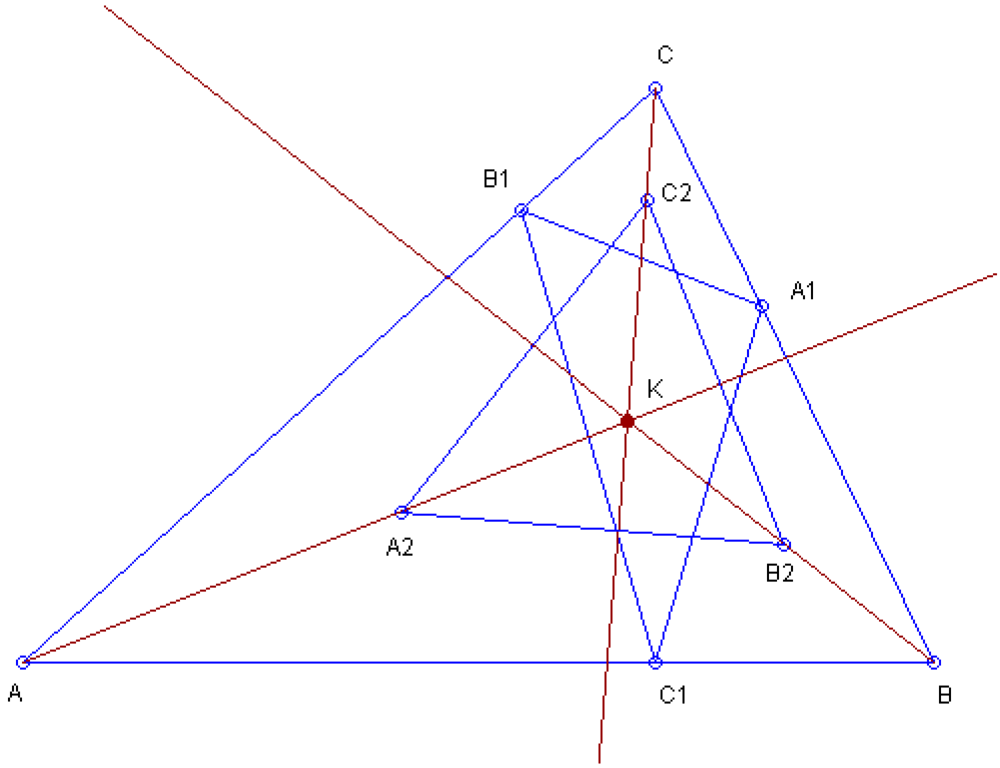
See the Figure:



K - Symmedian Point;
$A_1B_1C_1$ - Antipedal Triangle of the Circumcenter;
$A_2B_2C_2$ - Circumcevian Triangle of the Symmedian Point;
Lines A_1A_2, B_1B_2 and C_1C_2 concur at point K, that is, K is the Perspector of triangles $A_1B_1C_1$ and $A_2B_2C_2$, that is, the Symmedian Point is the Perspector of the Antipedal Triangle of the Circumcenter and the Circumcevian Triangle of the Symmedian Point.

Theorem 54. The Symmedian Point is the Perspector of Triangle ABC and the Triangle of the Centroids of the Corner Triangles of the Orthocenter.

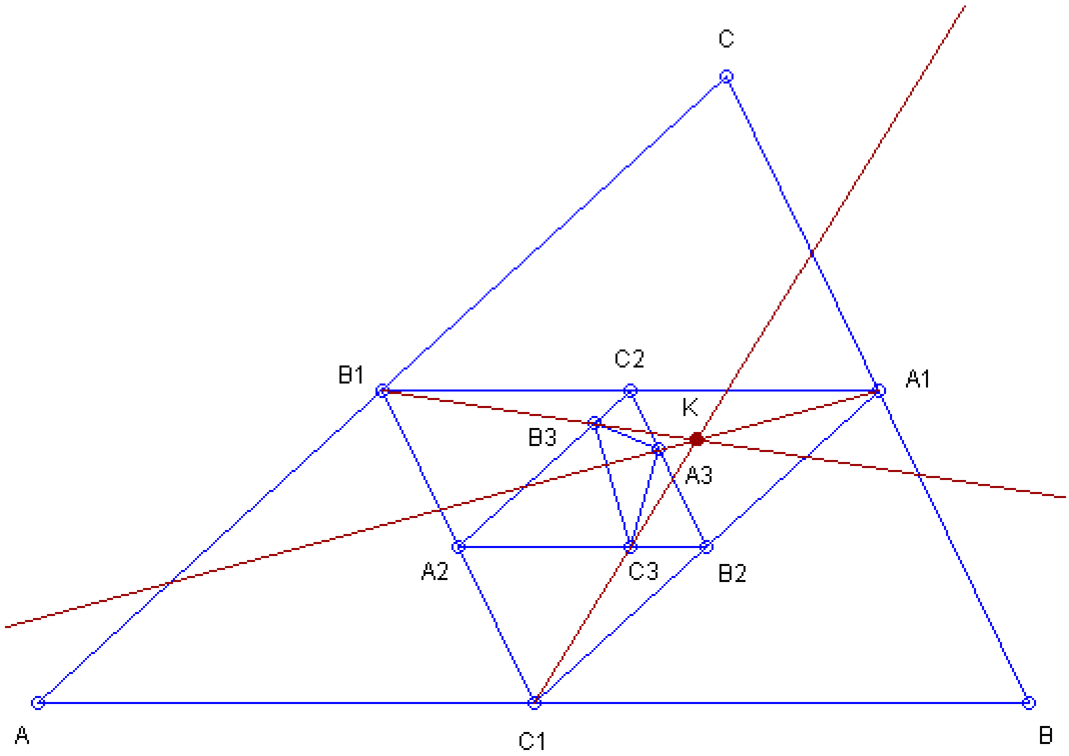
See the Figure:



K - Symmedian Point;
$A_1B_1C_1$ - Orthic Triangle = Cevian triangle of the Orthocenter;
A_2 - Centroid of triangle AB_1C_1;
B_2 - Centroid of triangle BC_1A_1;
C_2 - Centroid of triangle CA_1B_1;
$A_2B_2C_2$ - Triangle of the Centroids of the Corner Triangles of the Orthocenter;
Lines AA_2, BB_2 and CC_2 concur at point K, that is, K is the Perspector of triangles ABC and $A_2B_2C_2$, that is, the Symmedian Point is the Perspector of Triangle ABC and the Triangle of the Centroids of the Corner Triangles of the Orthocenter.

Theorem 104. The Symmedian Point is the Perspector of the Medial Triangle and the Orthic Triangle of the Medial Triangle of the Medial Triangle.

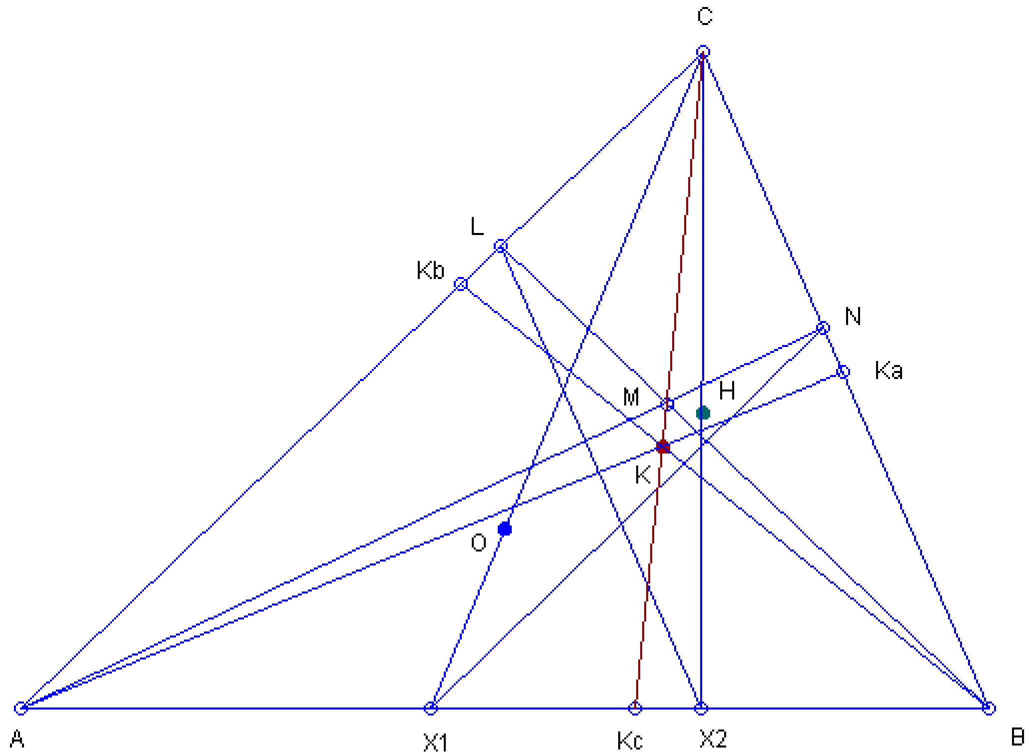
See the Figure:



K - Symmedian Point;
$A_1B_1C_1$ - Medial Triangle;
$A_2B_2C_2$ - Medial Triangle of the Medial Triangle;
$A_3B_3C_3$ - Orthic Triangle of the Medial Triangle of the Medial Triangle;
Lines A_1A_3, B_1B_3 and C_1C_3 concur at point K, that is, K is the Perspector of triangles $A_1B_1C_1$ and $A_3B_3C_3$, that is, the Symmedian Point is the Perspector of the Medial Triangle and the Orthic Triangle of the Medial Triangle of the Medial Triangle.

Theorem 105. The Symmedian Point is the Product of the Circumcenter and the Orthocenter.

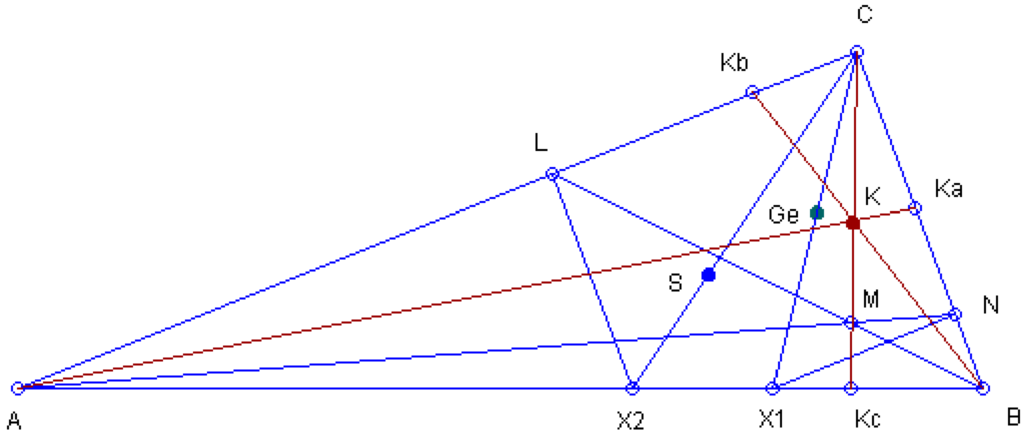
See the Figure:



O - Circumcenter;
H - Orthocenter;
K - Symmedian Point;
X_1 - intersection point of lines AB and CO;
X_2 - intersection point of lines AB and CH;
N - intersection point of line BC and the line through X_1 parallel to line AC;
L - intersection point of line AC and the line through X_2 parallel to line BC;
M - intersection point of lines AN and BL;
K_c - intersection point of lines AB and CM;
Similarly, construct points K_a and K_b;
K is the intersection point of lines AK_a, BK_b and CK_c, that is, the Symmedian Point is the Product of the Circumcenter and the Orthocenter.

Theorem 106. The Symmedian Point is the Product of the Gergonne Point and the Internal Center of Similitude of the Incircle and the Circumcircle.

See the Figure:



Ge - Gergonne Point;
S - Internal Center of Similitude of the Incircle and the Circumcircle;
K - Symmedian Point;
X_1 - intersection point of lines AB and CGe;
X_2 - intersection point of lines AB and CS;
N - intersection point of line BC and the line through X_1 parallel to line AC;
L - intersection point of line AC and the line through X_2 parallel to line BC;
M - intersection point of lines AN and BL;
K_c - intersection point of lines AB and CM;
Similarly, construct points K_a and K_b;
K is the intersection point of lines AK_a, BK_b and CK_c, that is, the Symmedian Point is the Product of the Circumcenter and the Orthocenter.

Theorems about the Symmedian Point and circles

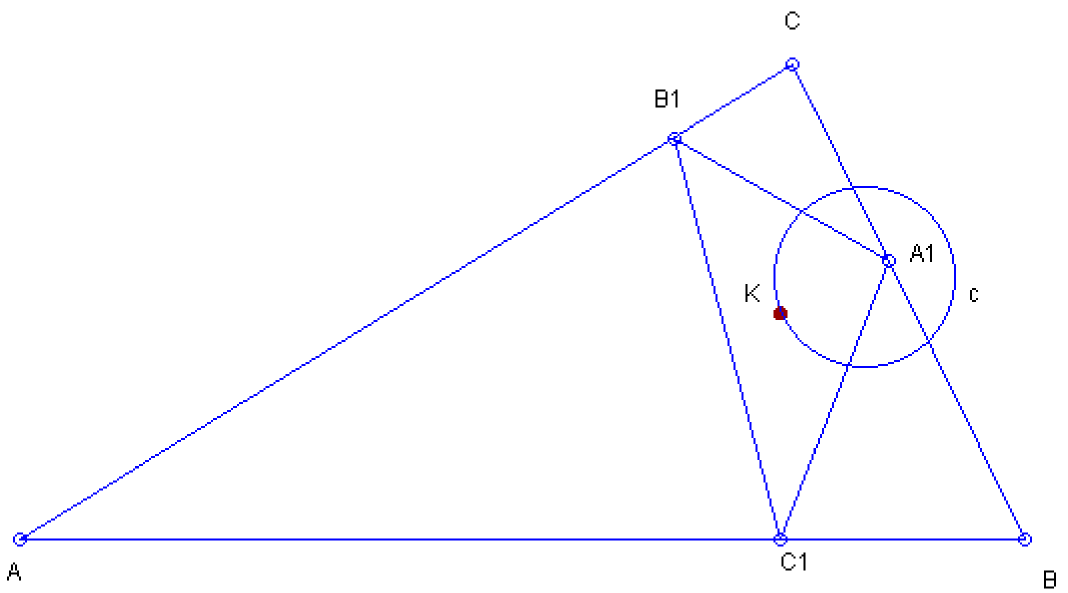
The list below contains a few theorems about the Symmedian Point, discovered by the Machine. Theorem 6 introduces two infinite families of circles passing through the Symmedian Point. Recall examples of named Tucker circles: Lemoine circle, Cosine Circle, Apollonius circle, Kenmotu circle, Gallatly circle, Taylor circle.

1. The Symmedian Point lies on the Orthocentroidal Circle of the Pedal Triangle of the Symmedian Point.
2. The Symmedian Point lies on the Parry Circle of the Pedal Triangle of the Symmedian Point.
3. The Symmedian Point lies on the Brocard Circle of the Pedal Triangle of the First Brocard Point.
4. The Symmedian Point lies on the Brocard Circle of the Pedal Triangle of the Second Brocard Point.
5. The Symmedian Point lies on the Brocard Circle of the Fourth Brocard Triangle.
6. For any Tucker Angle, the Symmedian Point lies on the Brocard Circles of the First and Second Tucker Triangles of the Tucker Angle.

We illustrate the above theorems. We invite the reader to select the new theorems and to prove them.

Theorem 1. The Symmedian Point lies on the Orthocentroidal Circle of the Pedal Triangle of the Symmedian Point.

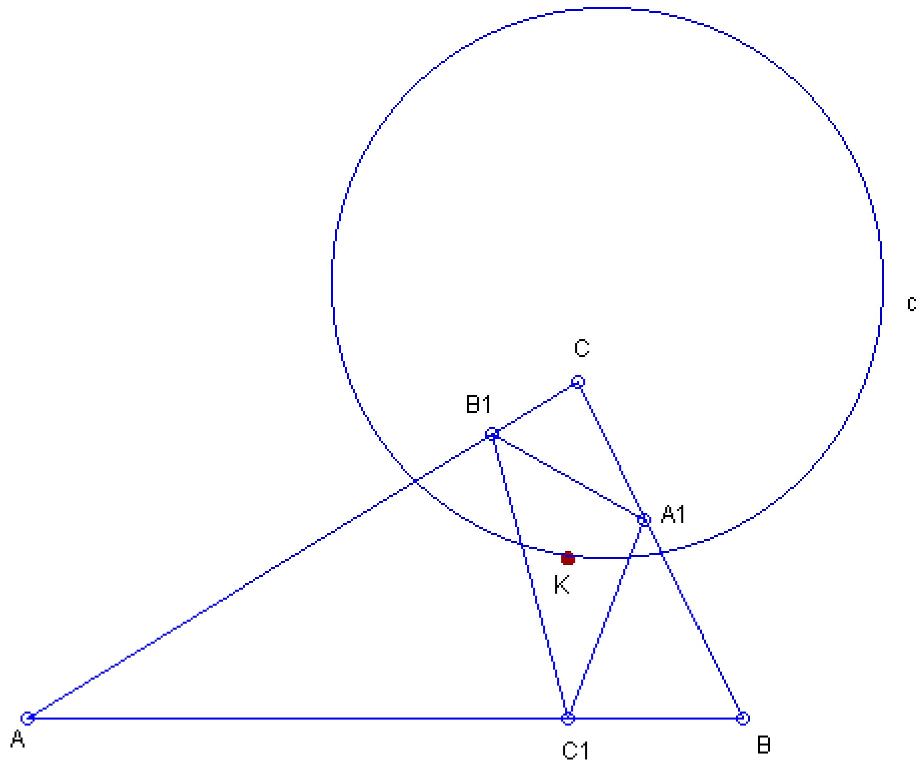
See the Figure:



K - Symmedian Point;
$A_1B_1C_1$ - Pedal Triangle of the Symmedian Point;
c - Orthocentroidal Circle of triangle $A_1B_1C_1$;
The Symmedian Point K lies on the Orthocentroidal Circle of the Pedal Triangle of the Symmedian Point.

Theorem 2. The Symmedian Point lies on the Parry Circle of the Pedal Triangle of the Symmedian Point.

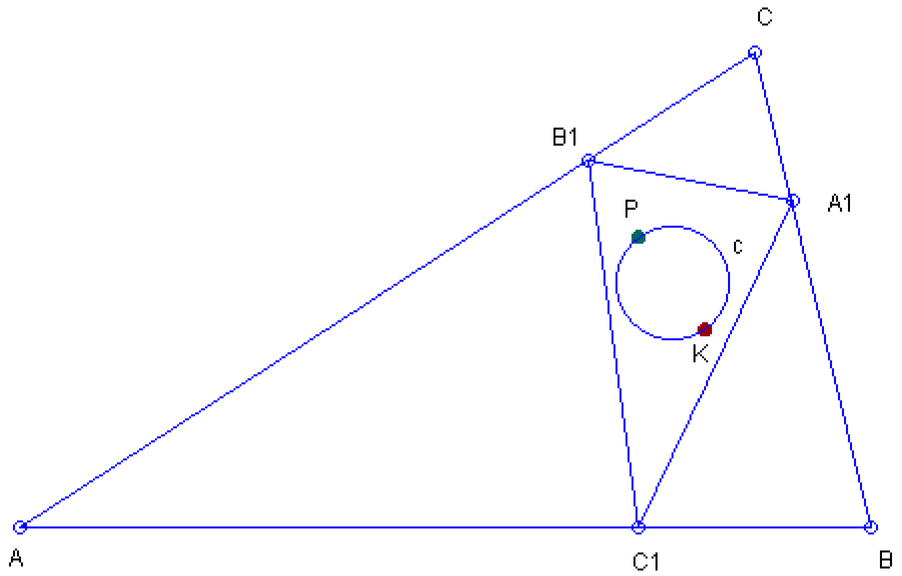
See the Figure:



K - Symmedian Point;
$A_1B_1C_1$ - Pedal Triangle of the Symmedian Point;
c - Parry Circle of triangle $A_1B_1C_1$;
The Symmedian Point K lies on the Parry Circle of the Pedal Triangle of the Symmedian Point.

Theorem 3. The Symmedian Point lies on the Brocard Circle of the Pedal Triangle of the First Brocard Point.

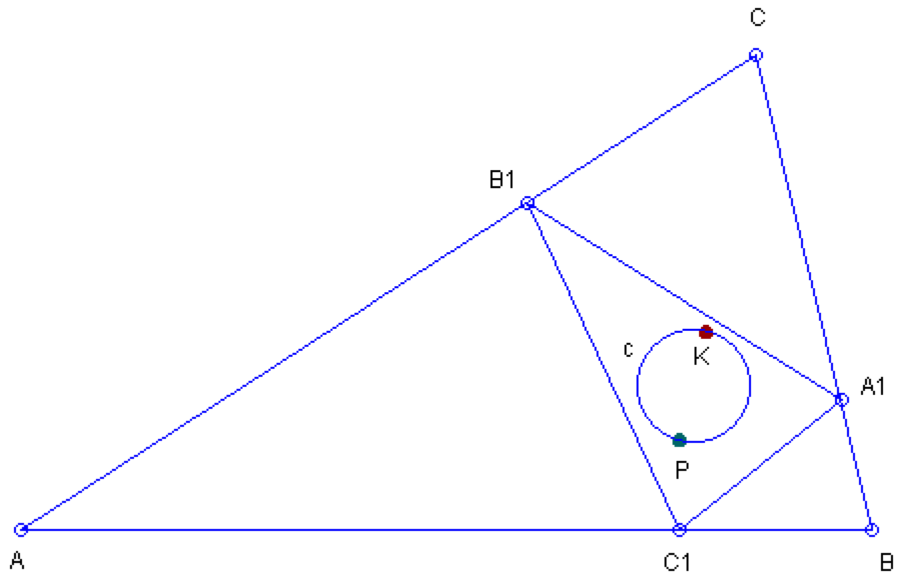
See the Figure:



K - Symmedian Point;
P - First Brocard Point;
$A_1B_1C_1$ - Pedal Triangle of the First Brocard Point;
c - Brocard Circle of triangle $A_1B_1C_1$;
The Symmedian Point K lies on the Brocard Circle of the Pedal Triangle of the First Brocard Point.

Theorem 4. The Symmedian Point lies on the Brocard Circle of the Pedal Triangle of the Second Brocard Point.

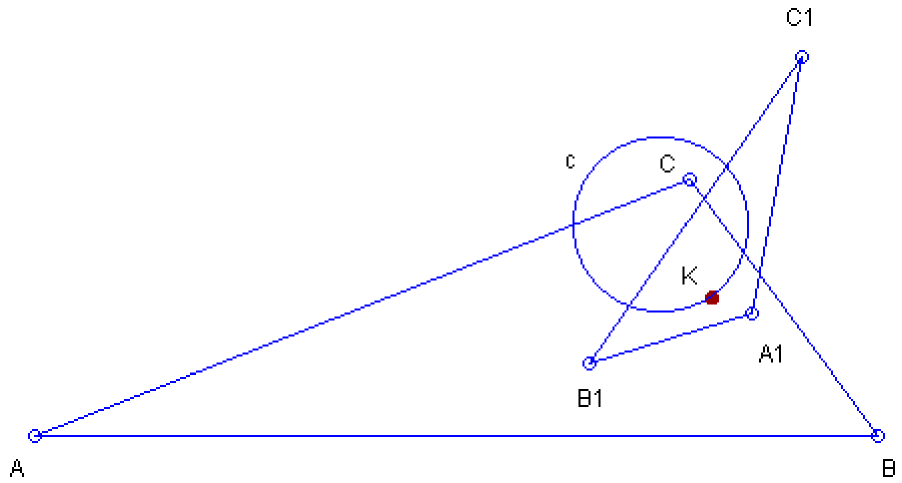
See the Figure:



K - Symmedian Point;
P - Second Brocard Point;
$A_1B_1C_1$ - Pedal Triangle of the Second Brocard Point;
c - Brocard Circle of triangle $A_1B_1C_1$;
The Symmedian Point K lies on the Brocard Circle of the Pedal Triangle of the Second Brocard Point.

Theorem 5. The Symmedian Point lies on the Brocard Circle of the Fourth Brocard Triangle.

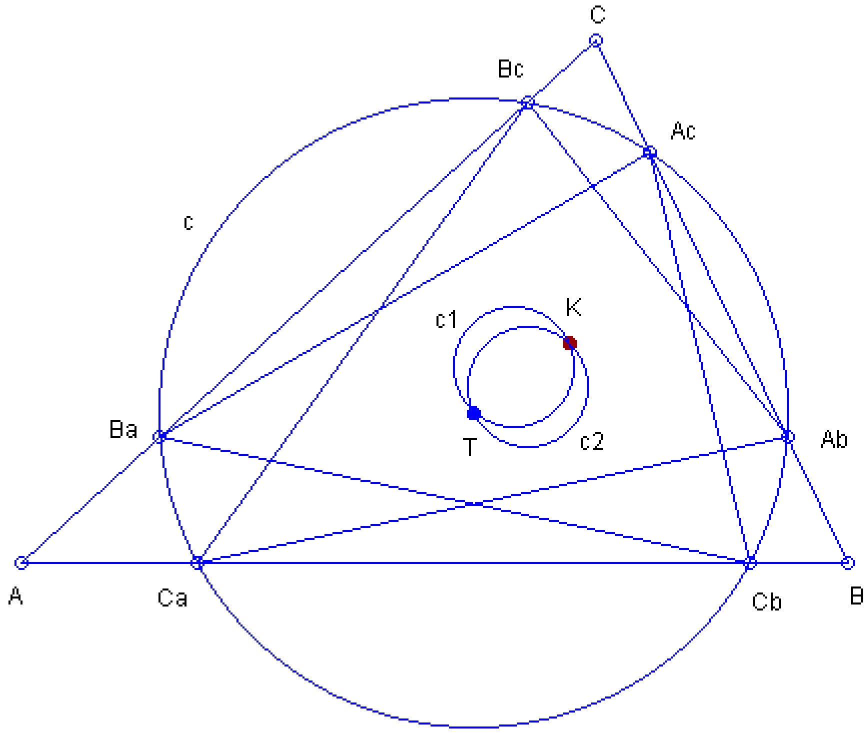
See the Figure:



K - Symmedian Point;
$A_1B_1C_1$ - Fourth Brocard Triangle;
c - Brocard Circle of triangle $A_1B_1C_1$;
The Symmedian Point K lies on the Brocard Circle of the Fourth Brocard Triangle.

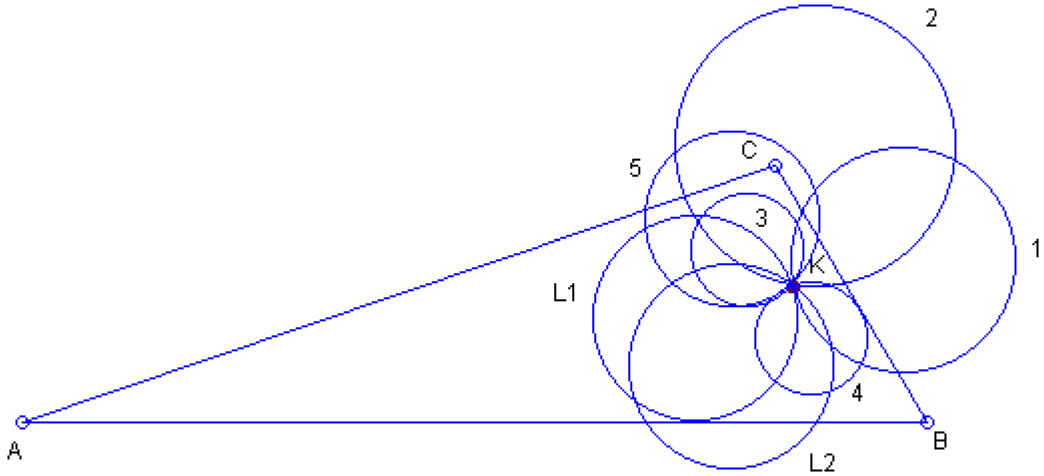
Theorem 6. For any Tucker Circle, the Symmedian Point lies on the Brocard Circles of the First and Second Tucker Triangles of the Tucker Circle.

See the Figure:



K - Symmedian Point;
c - Tucker Circle;
T - Center of the Tucker Circle;
AbBcCa - First Tucker Triangle of the Tucker Circle;
AcBaCb - Second Tucker Triangle of the Tucker Circle;
c1 - Brocard Circles of the First Tucker Triangle of the Tucker Circle;
c2 - Brocard Circles of the Second Tucker Triangle of the Tucker Circle;
The Symmedian Point K lies on the Brocard Circles of the First and Second Tucker Triangles of the Tucker Circle.

We illustrate the above theorems. See the Figure:



K - Symmedian Point;
The Symmedian Point lies on the following circles:
1 - Orthocentroidal Circle of the Pedal Triangle of the Symmedian Point;
2 - Parry Circle of the Pedal Triangle of the Symmedian Point;
3 - Brocard Circle of the Pedal Triangle of the First Brocard Point;
4 - Brocard Circle of the Pedal Triangle of the Second Brocard Point;
5 - Brocard Circle of the Fourth Brocard Triangle;
From the infinite series of circles defined in theorem 6, we take two circles:
L1 - Brocard Circle of the First Tucker Triangle of the Lemoine Circle;
L2 - Brocard Circle of the Second Tucker Triangle of the Lemoine Circle.

Theorems about Symmedian points of triangles

In different triangles the symmedian points of these triangles could coincide with other named points. E. Weisstein gave a table [4, Symmedian Point] to illustrate symmedian point of different triangles. Here we give a computer-generated theorems about the symmedian points of different triangles. Our theorems are an addition to the Weisstein' s table.

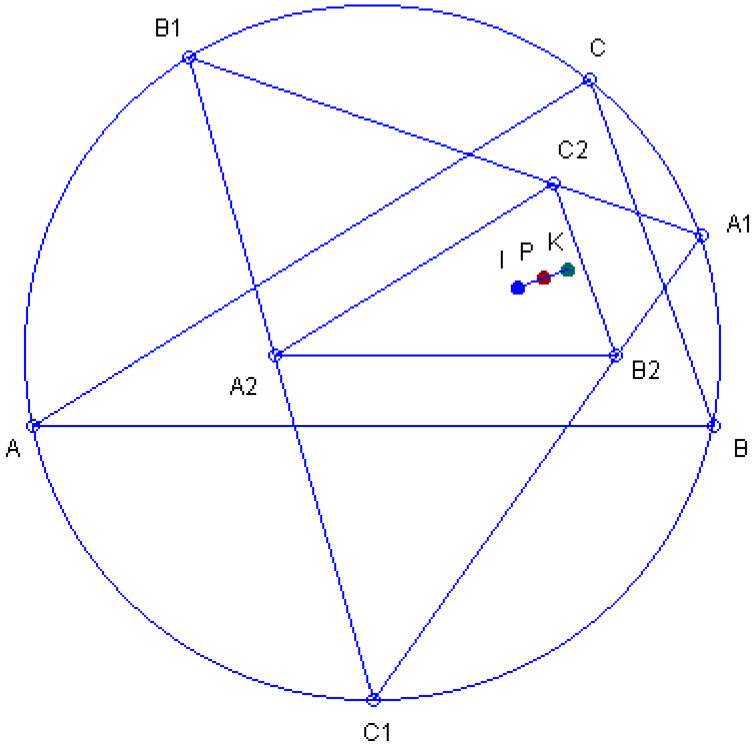
1. The Symmedian Point of the Intouch Triangle of the Medial Triangle is the Mittenpunkt.
2. The Symmedian Point of the Orthic Triangle of the Circum-Incentral Triangle is the Midpoint of the Incenter and the Symmedian Point.
3. The Symmedian Point of the Intouch Triangle of the Euler Triangle is the Midpoint of the Gergonne Point and the Orthocenter.
4. The Symmedian Point of the Excentral Triangle of the Medial Triangle is the Complement of the Mittenpunkt.
5. The Symmedian Point of the Excentral Triangle of the Anticomplementary Triangle is the Gergonne Point.
6. The Symmedian Point of the Anticomplementary Triangle of the Anticomplementary Triangle is the Perspector of the Orthic Triangle and the Anticomplementary Triangle.
7. The Symmedian Point of the Excentral Triangle of the Euler Triangle is the Midpoint of the Mittenpunkt and the Orthocenter.
8. The Symmedian Point of the Excentral Triangle of the Johnson Triangle is the

Midpoint of the Gergonne Point and the Orthocenter.
9. The Symmedian Point of the Circum-Incentral Triangle of the Anticomplementary Triangle is the Midpoint of the Gergonne Point and the Nagel Point.
10. The Symmedian Point of the Hexyl Triangle of the Medial Triangle is the Midpoint of the Mittenpunkt and the Orthocenter.
11. The Symmedian Point of the Johnson Triangle of the Medial Triangle is the Center of the Brocard Circle.
12. The Symmedian Point of the Fourth Brocard Triangle of the Intouch Triangle is the Gergonne Point.
13. The Symmedian Point of the Fourth Brocard Triangle of the Excentral Triangle is the Mittenpunkt.
14. The Symmedian Point of the Fourth Brocard Triangle of the Circum-Incentral Triangle is the Midpoint of the Incenter and the Mittenpunkt.
15. The Symmedian Point of the Fourth Brocard Triangle of the Euler Triangle is the Midpoint of the Orthocenter and the Symmedian Point.
16. The Symmedian Point of the Hexyl Triangle of the Euler Triangle is the Complement of the Mittenpunkt.
17. The Symmedian Point of the Fourth Brocard Triangle of the Fourth Brocard Triangle is the Symmedian Point.
18. The Symmedian Point of the Intouch Triangle of the Medial Triangle of the Medial Triangle is the Complement of the Mittenpunkt.
19. The Symmedian Point of the Intouch Triangle of the Orthic Triangle of the Circum-Incentral Triangle is the Midpoint of the Gergonne Point and the Incenter.
20. The Symmedian Point of the Intouch Triangle of the Medial Triangle of the Euler Triangle is the Midpoint of the Mittenpunkt and the Orthocenter.
21. The Symmedian Point of the Intouch Triangle of the Euler Triangle of the Medial Triangle is the Midpoint of the Circumcenter and the Mittenpunkt.
22. The Symmedian Point of the Excentral Triangle of the Anticomplementary Triangle of the Euler Triangle is the Midpoint of the Gergonne Point and the Orthocenter.
23. The Symmedian Point of the Anticomplementary Triangle of the Euler Triangle of the Medial Triangle is the Center of the Brocard Circle.

We illustrate a few of the above theorems. We invite the reader to select the new theorems and to prove them.

Theorem 2. The Symmedian Point of the Orthic Triangle of the Circum-Incentral Triangle is the Midpoint of the Incenter and the Symmedian Point.

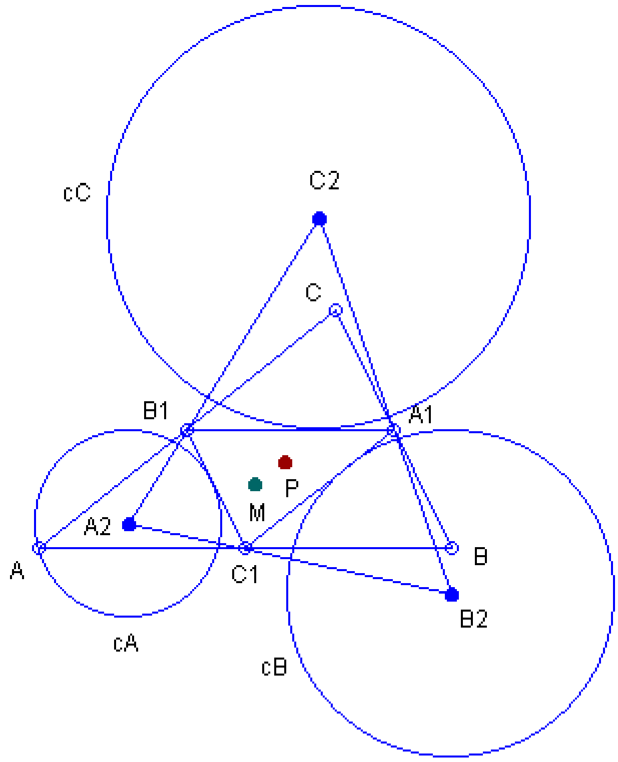
See the Figure:



I - Incenter;
K - Symmedian Point;
P - Midpoint of the Incenter and the Symmedian Point;
$A_1B_1C_1$ - Circum-Incentral Triangle;
$A_2B_2C_2$ - Orthic Triangle of the Circum-Incentral Triangle;
P is the Symmedian Point of triangle $A_2B_2C_2$, that is, the Symmedian Point of the Orthic Triangle of the Circum-Incentral Triangle is the Midpoint of the Incenter and the Symmedian Point.

Theorem 4. The Symmedian Point of the Excentral Triangle of the Medial Triangle is the Complement of the Mittenpunkt.

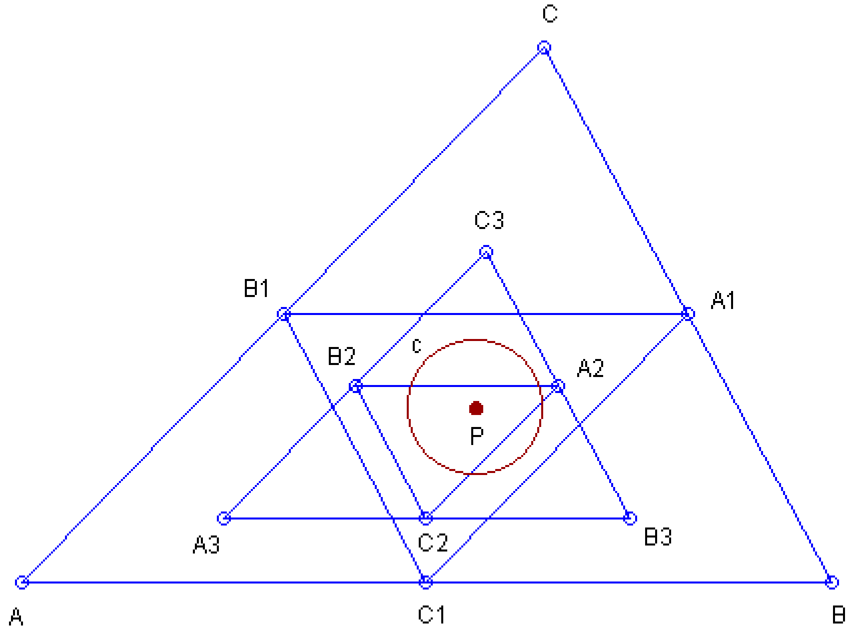
See the Figure:



M - Mittenpunkt;
P - Complement of the Mittenpunkt;
$A_1B_1C_1$ - Medial Triangle;
$A_2B_2C_2$ - Excentral Triangle of the Medial Triangle;
P is the Symmedian Point of triangle $A_2B_2C_2$, that is, the Symmedian Point of the Excentral Triangle of the Medial Triangle is the Complement of the Mittenpunkt.

. **Theorem 23.** The Symmedian Point of the Anticomplementary Triangle of the Euler Triangle of the Medial Triangle is the Center of the Brocard Circle.

See the Figure:



c - Brocard Circle;
P - Center of the Brocard Circle;
$A_1B_1C_1$ - Medial Triangle;
$A_2B_2C_2$ - Euler Triangle of the Medial Triangle;
$A_3B_3C_3$ - Anticomplementary Triangle of the Euler Triangle of the Medial Triangle;
P is the Symmedian Point of triangle $A_3B_3C_3$, that is, the Symmedian Point of the Anticomplementary Triangle of the Euler Triangle of the Medial Triangle is the Center of the Brocard Circle.

Thanks

The figures in this note are produced by using the program C.a.R. (Compass and Ruler), an amazing program created by Rene Grothmann. The Grothmann' s program is available for download at the Web. It is free and open source. Many thanks to Rene Grothmann for his wonderful program.

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