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COMPUTER DISCOVERED MATHEMATICS: A NOTE ON THE GRINBERG HOMOTHETY

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ABSTRACT. By using the computer program ”Discoverer” we study the Grinberg homothety.

1. INTRODUCTION

Let P be a point in the plane of triangle ABC. Denote by ctP the complement of the isotomic conjugate of P. Then the cevian triangle of P and the anticevian triangle of ctP are homothetic.

This result is published in 2003 by Darij Grinberg [3]. We call the homothety *Grinberg homothety of P* and we call the center of the Ginberg homothety of point P the *Grinberg point of point P*.

Note that in Kimberling [7] the Grinberg point is called the Danneels point of P (See the preambles to points X(3078) and X(8012) in [7]). In Douillet [2] the Grinberg point is called the First Danneels perspector. These names are because of the publication by Danneels from 2005 [1].

In this paper we find the ratio of the Grinberg homothety and we consider special cases of the point P. Also, we find the barycentric coordinates of a few new Grinberg points.

We use barycentric coordinates. See [4], [8], [10]. The side lengths of triangle ABC are denoted by $BC = a, CA = b$ and $AB = c$.

We use the computer program “Discoverer” created by the authors. See [5], [6].

The following theorem is published by Darij Grinberg in 2003 [3]:

Theorem 1.1. *The barycentric coordinates of the center of the Grinberg homothety of point $P = (u, v, w)$, that is, the Grinberg point of point P, are*

$$(u^2(v+w), v^2(w+u), w^2(u+v)). \tag{1.1}$$

2. THE RATIO OF THE GINBERG HOMOTHETY

Theorem 2.1. *The ratio k of the Grinberg homothety of point $P = (u, v, w)$ is*

$$k = \frac{(u+v)(v+w)(w+u)}{2uvw}. \tag{2.1}$$

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Proof. In order to find the ratio of the homothety from the cevian triangle $PaPbPc$ of $P = (u,v,w)$ to the anticevian triangle $QaQbQc$ of the complement of the isotomic conjugate of P it is enough to calculate the distances OPa and OQa where O is the center of the homothety. We use the distance formula (9) in [4].

The center of the homothety O has barycentric coordinates $(u^2(v+w), v^2(w+u), w^2(u+v))$ (See Theorem 1.1).

Vertex Pa has barycentric coordinates $(0, v.w)$.

Vertex Qa has barycentric coordinates $(-u(v+w), v(w+u), w(u+v)$.

For the ratio k we obtain $k = \dfrac{(u+v)(v+w)(w+u)}{2uvw}$. □

The "Discoverer" has investigated 195 Grinberg points. Of them 36 are Kimberling points [7] and the rest of 159 points are new points, not available in [7]. See the Supplementary material.

Today many Grinberg points are included in [7] (under the name Danneels points). (See the preamble to point X(8012))

Below we give examples of the ratios of a few Grinberg homotheties with known centers.

Example 1. The Grinberg point of the Incenter is the point X(42). The Ratio of the Grinberg homothety of the Incenter is $k = \dfrac{(a+b)(b+c)(c+a)}{2abc}$.

Example 2. The Grinberg point of the Centroid is the Centroid. The Ratio of the Grinberg homothety of the Centroid is $k = 4$.

Example 3. The Grinberg point of the Orthocenter is point X(25). The Ratio of the Grinberg homothety of the Orthocenter is

$$k = \frac{4a^2b^2c^2}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

If triangle ABC is acute, $k > 0$.

If triangle ABC is obtuse, $k < 0$.

Example 4. The Grinberg point of the Symmedian point is point X(3051). The Ratio of the Grinberg homothety of the Symmedian point is

$$k = \frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{2a^2b^2c^2}.$$

Example 5. The Grinberg point of the Gergonne point is point X(57). The Ratio of the Grinberg homothety of the Gergonne point is

$$k = \frac{4abc}{(b+c-a)(c+a-b)(a+b-c)}.$$

Example 6. The Grinberg point of the Nagel point is point X(200). The Ratio of the Grinberg homothety of the Nagel point is

$k = \dfrac{4abc}{(b+c-a)(c+a-b)(a+b-c)}$. (the same as the ratio in Example 5).

3. New Grinberg Points

In the Supplementary material we list 159 new Grinberg points, not available in [7]. We encourage the readers, by using (1) and (2), to find the barycentric coordinates of these new Grinberg points and the ratios of the corresponding Grinberg homotheties.

Below we give an example.

References about Theorem 3.1: Yff center of congruence in Wikipedia [9], Point X(174) in Kimberling [7].

Denote $E_1 = \sqrt{a(b+c-a)}, E_2 = \sqrt{b(c+a-b)}, E_3 = \sqrt{c(a+b-c)}$.

Theorem 3.1. *The Grinberg Point of the Yff Center of Congruence has barycentric coordinates*

$$Gr(X(174)) = \left(a^2E_2E_3(bE_3+cE_2), b^2E_3E_1(cE_1+aE_3), c^2E_1E_2(aE_2+bE_1)\right).$$

The ratio of the Grinberg homothety of the Yff Center of Congruence is

$$k = \frac{(cE_1+aE_3)(aE_2+bE_1)(bE_3+cE_2)}{2abcE_1E_2E_3}$$

Proof. The Yff Center of Congruence has barycentric coordinates $(aE_2E_3, bE_3E_1, cE_1E_2)$. We use formulas (1) and (2). ☐

By using the "Discoverer" we could find properties of the new Grinberg points. Below is an example.

Theorem 3.2. *The Grinberg Point of the X(174) Yff Center of Conguence is the Cross Product (Crossmul in [2]) of point X(174) and point X(57) Isogonal Conjugate of the Mittenpunkt.*

Proof. The barycentric coordinates of points X(174) and X(57) are known. (See e.g. [7]). The barycentric coordinates of the Cross Product of X(174)=(u, v, w) and X(57)=(x, y, z) are as follows (See e.g. the crossmul formula (4.7) in Section 4.8, Douillet, [2]):

$$P = ((vz+wy)ux, (wx+uz)vy, (uy+vx)wz).$$

The barycentric coordinates of the Grinberg point of X(174) and point P coincide. This completes the proof. ☐

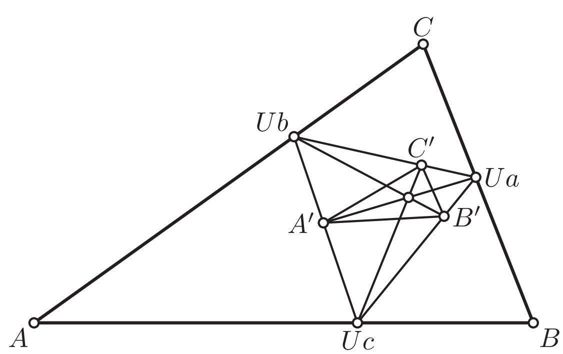

FIGURE 1

Figure 1 illustrates Theorem 3.2. Denote by X is the Isogonal conjugate of the Mittenpunkt. In figure 1,

- $UaUbUc$ is the Cevian triangle of the Yff center of congruence,
- A' is the point of intersection of lines AX and $UbUc$,
- B' is the point of intersection of lines BX and $UcUa$,
- C' is the point of intersection of lines CX and $UaUb$.

Then lines $A'Ua, B'Ub$ and $C'Uc$ concur at the Grinberg point of the Yff center of congruence.

Note that the Grinberg point of point P always lies on the line GP through an arbitrary point P and the Centroid G, as it is stated without proof in the preamble of point X(8012), [7]. Below we give the proof. Denote $P = (u, v, w)$. By using formula (3) in [4], we find the equation of the line $L = GP$ as follows:

$$L : (v - w)x + (w - u)y + (u - v)z = 0.$$

Now in this equation we substitute x, y, z for the barycentric coordinates of the Grinberg point of P, given in (1), and we obtain 0=0. This completes the proof.

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

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$http://car.rene-grothmann.de/doc\_en/index.html.$
See also:
$http://www.journal-1.eu/2016-1/Grothmann-CaR-pp.45-61.pdf.$

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