102.13 Distance from the incentre of the tangential triangle of an obtuse triangle to the Euler line

The tangential triangle of a triangle *ABC* is the triangle formed by the lines tangent to the circumcircle of the given triangle *ABC* at its vertices. If *ABC* is acute, then its circumcircle is the incircle of its tangential circle. It follows that the incentre of its tangential triangle is the circumcentre *O* of *ABC*, which lies on its Euler line. If, however, *ABC* is obtuse, then its circumcircle is an excircle of the tangential circle, and *O* is one of its excentres.

In this Note we calculate the distance from the incentre of the tangential triangle of an obtuse triangle *ABC* to the Euler line of the triangle *ABC*.

Theorem 1: Given an obtuse triangle *ABC* with side lengths $BC = a$, $CA = b$ and $AB = c$ such that $c > a$ and $c > b$. Then the distance d from the incentre of the tangential triangle of triangle *ABC* to the Euler line of triangle *ABC* is as follows:

$$d = \frac{2a^2b^2c^2\left|a^2 - b^2\right|}{\left(b^2 + c^2 - a^2\right)\left(c^2 + a^2 - b^2\right)\sqrt{R}},$$

where

$$R = \left|a^6 + b^6 + c^6 + 3a^2b^2c^2 - a^4b^2 - a^2b^4 - b^4c^2 - b^2c^4 - c^4a^2 - c^2a^4\right|.$$

Proof: We use barycentric coordinates. We refer the reader to [1, 2]. Let *ABC* be an obtuse triangle such that $c > a$ and $c > b$. By using formula (3) in [1] we find the equation of the Euler line L as the line joining the centroid $G\,(1,\ 1,\ 1)$ to the circumcentre

$$O\left(a^2\left(b^2 + c^2 - a^2\right), b^2\left(c^2 + a^2 - b^2\right), c^2\left(a^2 + b^2 - c^2\right)\right)$$

as follows:

$$L : \left(b^2 - c^2\right)\left(b^2 + c^2 - a^2\right)x + \left(c^2 - a^2\right)\left(c^2 + a^2 - b^2\right)y$$
$$+\left(a^2 - b^2\right)\left(a^2 + b^2 - c^2\right)z = 0.$$

The barycentric coordinates of the tangential triangle $T = T_AT_BT_C$ of triangle *ABC* are as follows [2, p. 54]:

$$T_A = \left(-a^2, b^2, c^2\right), \qquad T_B = \left(a^2, -b^2, c^2\right), \qquad T_C = \left(a^2, b^2, -c^2\right),$$

The side-lengths of triangle T are as follows (formulas (2)-(4) in [3]):

$$a_T = \frac{2a^3bc}{\left|a^4 - \left(b^2 - c^2\right)^2\right|}, \qquad b_T = \frac{2ab^3c}{\left|b^4 - \left(c^2 - a^2\right)^2\right|}, \qquad c_T = \frac{2abc^3}{\left|c^4 - \left(a^2 - b^2\right)^2\right|}.$$

Since $c^2 > a^2 + b^2$, we take the following positive values of the side-lengths:

$$a_T = \frac{2a^3bc}{\left(a^2 - b^2 + c^2\right)\left(c^2 - a^2 - b^2\right)},$$

$$b_T = \frac{2ab^3c}{(c^2 - a^2 - b^2)(b^2 - a^2 + c^2)},$$

$$c_T = \frac{2abc^3}{(b^2 - a^2 + c^2)(a^2 - b^2 + c^2)}.$$

Then the barycentric coordinates of the incentre I_T with respect to triangle T are as follows: $I_T = (a_T, b_T, c_T)$ equivalently:

$$I_T = \left(a^2(b^2 + c^2 - a^2), b^2(a^2 - b^2 + c^2), c^2(c^2 - a^2 - b^2)\right).$$

Now, by using formula (10) in [1] we obtain the barycentric coordinates of I_T with respect to triangle ABC as follows:

$$I_T = \left(a^2(a^2 - b^2 + c^2), b^2(b^2 - a^2 + c^2), -c^2(a^2 + b^2 + c^2)\right).$$

By using the barycentric coordinates of I_T with respect to triangle ABC and formula (8) in [1], we find the equation of the line L_1 passing through I_T and perpendicular to the Euler line L.

Then we use formula (5) in [1] and we find the intersection Q of lines L_1 and L.

Finally, we use formula (9) in [1] and we find the distance between points I_T and Q, that is, the distance from point I_T to the Euler line. This completes the proof.

If we want to avoid calculations by hand, we may use a computer algebra system like *Maple* or *Mathematica*.

If $a = b$ in the statement of Theorem 1, then the incentre of the tangential triangle lies on the Euler line.

Notice that in the above theorem another way of finding the barycentric coordinates of the incentre I_T is by using formula (17) in [1]. Indeed, the orthic and tangential triangles are homothetic, [4], so that we can use the homothety to find the incentre I_T as the homothetic image of the incentre of the orthic triangle. The centre of the homothety is the point X(25) in [5], the scale factor k of the homothety is

$$k = \frac{4a^2b^2c^2}{(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)}.$$

and the incentre of the orthic triangle is the vertex C of triangle ABC (provided triangle ABC is obtuse and $c = AB$ is the longest side). Now formula (17) in [1] gives the barycentric coordinates of I_r.

We leave to the reader the following theorem:

Theorem 2: Given an obtuse triangle ABC with side-lengths $BC = a$, $CA = b$ and $AB = c$ such that $c > a$ and $c > b$. Then the distance d from the incentre of the tangential triangle of triangle ABC to the circumcentre of triangle ABC is

$$d = \frac{a^2b^2c^3}{(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)\Delta}$$

where Δ is the area of triangle ABC.

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