102.14 A Note on the Feuerbach triangle

The Feuerbach triangle is the triangle formed by the three points of tangency of the nine-point circle with the excircles (see [1, 2, 3]).

In this Note we present some results about the Feuerbach triangle. We use barycentric coordinates. We encourage the reader to find synthetic proofs of the theorems.

The side-lengths of the reference triangle ABC are denoted by $a = BC$, $b = CA$, $c = AB$. The area of triangle ABC is denoted by Δ. We denote the Feuerbach triangle by $F = F_aF_bF_c$.

Theorem 1: The barycentric coordinates of the Feuerbach triangle are as follows:

$$F_a = \left(-(b-c)^2(a+b+c), (a+c)^2(a+b-c), (a+b)^2(a-b+c)\right),$$

$$F_b = \left((b+c)^2(a+b-c), -(a-c)^2(a+b+c), (a+b)^2(b+c-a)\right),$$

$$F_c = \left((b+c)^2(a-b+c), (a+c)^2(b+c-a), -(a-b)^2(a+b+c)\right).$$

Proof: The vertex F_a is the internal centre of similitude of the A-excircle and the nine-point circle. We use the formula for the internal similitude centre of two circles. (See [4].) Similarly for vertices F_b and F_c.

Theorem 2: The area of the Feuerbach triangle is

$$\frac{2a^2b^2c^2(b+c)(c+a)(a+b)\Delta}{E_1E_2E_3},$$

where

$$E_1 = a^2b - ab^2 + a^2c - ac^2 + b^2c + bc^2 + 3abc + a^3 - b^3 - c^3,$$

$$E_2 = -a^2b + ab^2 + a^2c + ac^2 + b^2c - bc^2 + 3abc - a^3 + b^3 - c^3,$$

$$E_3 = a^2b + ab^2 - a^2c + ac^2 - b^2c + bc^2 + 3abc - a^3 - b^3 + c^3.$$

Proof: We use Theorem 1 and the area formula (2) in [4].

Theorem 3: The side-lengths of the Feuerbach triangle are as follows:

$$a_F = F_bF_c = \frac{abc(b+c)}{\sqrt{E_2E_3}},$$

$$b_F = F_cF_a = \frac{abc(c+a)}{\sqrt{E_3E_1}},$$

$$c_F = F_aF_b = \frac{abc(a+b)}{\sqrt{E_1E_2}},$$

where E_1, E_2, E_3 are as in Theorem 2.

Proof: We use the distance formula (9) in [4].

See also S. Kiss, Theorem 3 in [5].

The Feuerbach triangle $F_aF_bF_c$ is in perspective with triangle ABC ([1, p. 305]) and the perspector is known as the Feuerbach perspector. (Point J^* in [1, Chap.21], point X(12) in [6]).

Theorem 4: The distance between the Feuerbach point F and the Feuerbach perspector P is

$$FP = \frac{4abc\Delta}{(a+b+c)\left(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 - abc - a^3 - b^3 - c^3\right)}.$$

Proof: We use the distance formula (9) in [4].

We leave the last two theorems as exercises to the reader. These theorems were discovered by the computer program ‘Discoverer’, created by the authors.

Triangle ABC is in perspective with the medial triangle of the incentral triangle of ABC, and the perspector is the *Grinberg point*. This is point X(37) in [6].

Denote by $A_PB_PC_P$ the cevian triangle of a point P [1, p.72]. Let A^P be the harmonic conjugate of P with respect to A and A_P. Define B^P and C^P cyclically. Triangle $A^PB^PC^P$ is the *anticevian triangle of point P*.

Theorem 5: The Feuerbach triangle is similar (but not homothetic) with the anticevian triangle of the Grinberg point. The ratio of similarity is

$$k = \frac{2abc\sqrt{abc}}{\sqrt{E_1E_2E_3}}$$

where E_1, E_2, E_3 are as in Theorem 2.

Reflect A_P, B_P, C_P, the vertices of the cevian triangle of point P, about the midpoints of sides BC, CA, AB, respectively, to obtain points A', B', C'. Then lines AA', BB', CC' concur in the *isotomic conjugate* of P [1, p. 82].

Let A^*, B^*, C^* be the midpoints of AA_P, BB_P, CC_P, respectively. Then $A^*B^*C^*$ is the *half-cevian triangle of ABC* [7].

Theorem 6: The Feuerbach triangle is similar (but not homothetic) with the half-cevian triangle of the isotomic conjugate of the incentre. The ratio of similarity is

$$k = \frac{2\sqrt{abc}\,(b+c)(c+a)(a+b)}{\sqrt{E_1E_2E_3}}$$

where E_1, E_2, E_3 are as in Theorem 2.

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